Math 220 - Calculus f. Business and Management - Worksheet 7

Solutions for Worksheet 7 - he number e and one-time investments

Periodic Compounding

Exercise 1a: *Chen has received* $\$5,000$ *as a gift from his grandfather. He puts the money in the bank where an annual interest rate of* 1.25% *is compounded monthly. How much will his investment be worth at the end of* 6 *years?*

Solution to #1a:

Gift amount $\rightsquigarrow$ $PV = \$5,000$
Annual interest rate $\rightsquigarrow$ $r = 1.25\%$
Monthly compounding $\rightsquigarrow$ $n = 12$
End of 6 *years* $\rightsquigarrow$ $t = 6$
Task: compute future worth of investment $\rightsquigarrow FV = PV\left(1 + r/n\right)^{n\cdot t} = 5,000\left(1 + .0125/12\right)^{72}$.

With a calculator or a spreadsheet you obtain $PV = \$5,389.21$.

Exercise 1b: Helen is saving to buy a car. Her bank is offering Certificates of Deposit (CDs) that pay a rate of 2.75% *compounded semi-annually if the money is left for* 5 *years. How much must Helen spend on CDs in order to have* $\$8,500$ *at the end of that time?*

Solution to #1b:

Annual interest rate $\rightsquigarrow$ $r = 2.75\%$
Semi-annual compounding $\rightsquigarrow$ $n = 2$
End of 5 *years* $\rightsquigarrow$ $t = 5$
Investment value after 5 *years* $\rightsquigarrow$ $FV = \$8,500$
Task: compute present amount of investment necessary $\rightsquigarrow PV = FV\,/\left(1 + r/n\right)^{n\cdot t} = 8,500/\left(1 + .0275/2\right)^{10}$.

With a calculator or a spreadsheet you obtain $PV = \$7,414.98$.

Exercise 1c: Laquisha needs $\$40,000$ *as a down payment for a house. She plans to put* $\$35,000$ *in a savings account that pays an annual interest rate of* 1.75% *quarterly. How long must she leave the money in order to reach her goal?*

Solution to #1c:

"needs for a downpayment" $\rightsquigarrow$ $FV = \$40,000$
"plans to invest $\rightsquigarrow$ $PV = \$35,000$
Annual interest rate $\rightsquigarrow$ $r = 1.75\%$
Quarter-annual compounding $\rightsquigarrow$ $n = 4$
"How long ..." $\rightsquigarrow$ t *Task: compute the investment horizon* t*, i.e., we must solve the following for* t*:*

$$\begin{aligned} FV &= PV\big(1 + r/n\big)^{nt} \\ \$40,000 &= \$35,000\big(1 + 0.0175/4\big)^{4t} \\ 40,000/35,000 &= 8/7 = \big(1 + 0.0175/4\big)^{4t} \\ \ln(8/7) &= \ln\Big(\big(1 + 0.0175/4\big)^{4t}\Big) = 4t\ln\big(1 + 0.0175/4\big) \\ t &= \frac{\ln(8/7)}{4\ln\big(1 + 0.0175/4\big)}. \end{aligned}$$

With a calculator or a spreadsheet you obtain $t = 7.647$ *years.*

Continuous Compounding

Exercise 2a: Chen has received a gift from his grandfather in height of $\$5,000$*. He puts it in the bank where an annual interest rate of* 1.25% *is compounded continuously. How much will his investment be worth at the end of* 6 *years?*

Solution to #2a:

Gift amount $\rightsquigarrow\ PV = \$5,000$
Annual interest rate $\rightsquigarrow\ r = 1.25\%$
End of 6 *years* $\rightsquigarrow\ t = 6$
Continuous compounding $\rightsquigarrow\ n =$ *N/A*
Task: compute future worth of investment $\rightsquigarrow FV = PV\, e^{r\cdot t} = 5,000\, e^{.0125\cdot 6}$.

With a calculator or a spreadsheet you obtain $PV = \$5,389.42$.

Exercise 2b: Helen is saving to buy a car. Her bank is offering Certificates of Deposit (CDs) that pay a rate of 2.75% *compounded continuously if the money is left for* 5 *years. How much must Helen spend on CDs in order to have* $\$8,500$ *at the end of that time?*

Solution to #2b:

Annual interest rate $\rightsquigarrow\ r = 2.75\%$
End of 5 *years* $\rightsquigarrow\ t = 5$
Investment value after 5 *years* $\rightsquigarrow\ FV = \$8,500$
Continuous compounding $\rightsquigarrow\ n =$ *N/A*
Task: compute present amount of investment necessary $\rightsquigarrow PV = FV\,/\,e^{r\cdot t} = 8,500/e^{5\cdot 0.0275}$.

With a calculator or a spreadsheet you obtain $PV = \$7,408.04$.

Exercise 2c: Laquisha needs $\$40,000$ *as a down payment for a house. She plans to put* $\$35,000$ *in a savings account that pays an annual interest rate of* 1.75% *compounded continuously. How long must she leave the money in order to reach her goal?*

Solution to #2c:

"needs for a downpayment" $\rightsquigarrow\ FV = \$40,000$
"plans to invest $\rightsquigarrow\ PV = \$35,000$
Annual interest rate $\rightsquigarrow\ r = 1.75\%$
Continuous compounding $\rightsquigarrow\ n =$ *N/A*
"How long ..." $\rightsquigarrow t$
Task: compute the investment horizon t*, i.e., we must solve the following for* t*:*

$$\begin{aligned} FV &= PVe^{rt} \\ \$40,000 &= \$35,000e^{.0175t} \\ 40,000/35,000 &= 8/7 = e^{.0175t} \\ \ln(8/7) &= \ln\left(e^{.0175t}\right) = .0175t \\ t &= \ln(8/7)/0.0175. \end{aligned}$$

With a calculator or a spreadsheet you obtain $t = 7.630$ *years.*

Effective Interest Rate

Exercise 3: Find the effective interest rate for each of the three problems in the two sections above on periodic and continuous compounding.

***Solutions to periodic compounding**:*

If r denotes the annual interest rate in an account that compounds interest n times per year, the effective interest rate is

$$\left(1+\frac{r}{n}\right)^n - 1$$

So, the effective interest rate of Chen's account is

$$\left(1+\frac{0.0125}{12}\right)^{12} - 1 \approx 0.01257186$$

The effective interest rate of Helen's account is

$$\left(1+\frac{0.0275}{2}\right)^{2} - 1 = 0.0276890625,$$

The effective interest rate of Laquisha's account is

$$\left(1+\frac{0.0175}{4}\right)^{4} - 1 \approx 0.017615179.$$

***Solutions to continuous compounding**:*

If r denotes the annual interest rate in an account that compounds interest continuously, the effective interest rate is $e^r - 1$.

So, the effective interest rate of Chen's account is $e^{0.0125} - 1 \approx 0.01257845$.

The effective interest rate of Helen's account is $e^{0.0275} - 1 \approx 0.02788161$.

The effective interest rate of Laquisha's account is $e^{0.0175} - 1 \approx 0.01765402$.

Random Word Problems

the solutions for all eight word problems can be found below the formulation of all those exercises.

Exercise D1: What is the effective interest rate for an investment that pays 6% per year compounded weekly (52 weeks/year)?

Exercise D2: How much will an investment of $\$1,250$ be worth if it is invested for 7 years at 8% compounded continuously?

Exercise D3: How long will it take for an investment of $\$500$ at 2.5% compounded monthly to be worth $\$575$?

Exercise D4: How much must be invested for four years at 4.5% compounded quarterly to be worth $\$625$ at the end of that time?

Exercise D5: What is the effective interest rate for an investment that pays 3.5% per year compounded continuously?

Exercise D6: How much must be invested for ten years at 2.5% compounded continuously to be worth $\$12,500$ at the end of that time?

Exercise D7: How much will an investment of $\$1,250$ be worth if it is invested for 7 years at 8% compounded semi-annually?

Exercise D8: How long will it take for an investment of $\$8,500$ at 2.25% compounded continuously to be worth $\$9,500$?

SOLUTIONS: *the solutions for all eight exercises above can be found in the following image that was taken from an Excel spreadsheet. Here are some explanations.*

The topmost row shows which problem is addressed in the resp. column.
The leftmost column indicates how the values in the grid were computed and you should be able to see how they relate to the formulas.
The formulas in the course material will usually express the annual interest rate as a decimal rather than a percentage: r *is usually represented, e.g., as* $.045$ *or* 0.045 *rather than* 4.5%*. In the spreadsheet image you will find the percentage notation.*
When compounding interest continuously there is no use for the value n *which indicates for periodically compounded interest how often interest is compounded per year. Accordingly, the value "N/A" is displayed in the "*$n =$ *frequency" row. In all other cases a cell is left blank (rather than zero) if it serves no purpose for computing the result.*

Problem in worksheet 7	wks-7, D1	wks-7, D2	wks-7, D3	wks-7, D4	wks-7, D5	wks-7, D6	wks-7, D7	wks-7, D8
r = interest rate	6.000%	8.000%	2.500%	4.500%	3.500%	2.500%	8.000%	2.250%
n = frequency	52	N/A	12	4	N/A	N/A	2	N/A
t = investment horizon		7	5.596299	4		10	7	4.943362
PV = principal value		$1,250.00	$500.00	$522.57		$9,735.01	$1,250.00	$8,500.00
FV = future value		$2,188.34	$575.00	$625.00		$12,500.00	$2,164.60	$9,500.00
(1 + r/n)^n = 1 + eff.int. rate	1.0618							
(1 + r/n)^(nt)				1.196015			1.731676448	
ln(FV/PV)			0.139762					
n * ln(1 + r /n)			0.024974					
e^r = 1 + eff.int. rate					1.03561971			
e^(rt)		1.7506725				1.284025417		
ln(FV/PV)								0.111226
eff.int. rate	6.180%				3.562%			

Figure 1: Excel data for random word problems