Math 220 - Calculus f. Business and Mgmt - Worksheet 9

Solutions for Worksheet 9 - Piecewise Defined Functions and Continuity

Evaluating and Graphing Functions

Exercise 1: *Compose these pairs of functions into a single function*

$$\textbf{\textit{1a}} : f(x) \;=\; \begin{cases} 2x+3 & \textit{if}\, x < 3, \\ x^2 & \textit{if}\, x \geq 3. \end{cases} \qquad \textit{Evaluate f(1), f(3), f(5) and graph the function.}$$

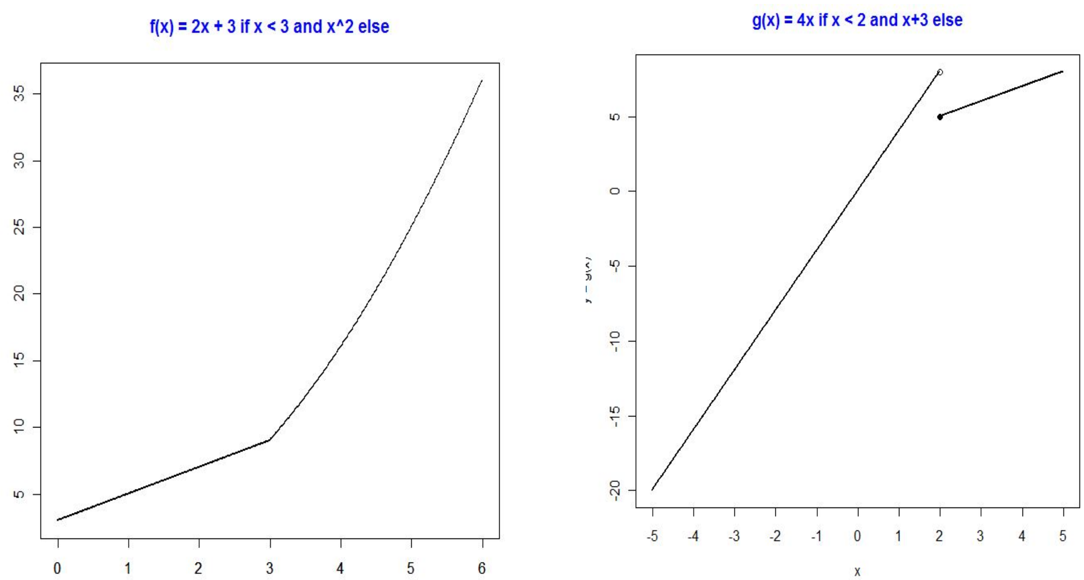
$$\textbf{\textit{1b}} : g(x) \;=\; \begin{cases} 4x & \textit{if}\, x < 2, \\ x+3 & \textit{if}\, x \geq 2. \end{cases} \qquad \textit{Evaluate g(-3), g(2), g(4) and graph the function.}$$

$$\textbf{\textit{1c}} : h(x) \;=\; \begin{cases} 3x+1 & \textit{if}\, x < 1, \\ x+3 & \textit{if}\, x > 1. \end{cases} \qquad \textit{Evaluate h(-3), h(2), h(4) and graph the function.}$$

Solution to #1:

The solutions to all three functions are given by means of Excel graphing sheets. Not the best tool available but it gets a good picture of the shape of the graph of each of the three functions.

As far as the evaluation $f(1), f(3), \cdots, g(-3), \cdots, h(1), h(3)$ *is concerned, just determine for each argument whether it belongs to the upper or lower branch. Note though that* $h(1)$ *is UNDEFINED because* 1 *does not belong to the domain* D_h *of* h*! See Figure 1: "Problems A1, A2: Graphs of f(x) and g(x)" and Figure 2: "Problem A3: Graph of h(x)" further down.*



(a) Graph of f(x) (b) Graph of g(x)

Figure 1: Problems 1a, 1b: Graphs of f(x) and g(x)

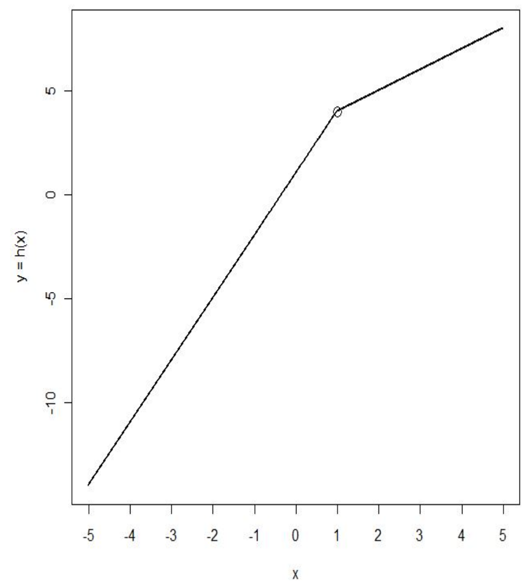


Figure 2: Problem 1c: Graph of h(x)

Building piecewise functions

Exercise 2: *Labor costs \$10 per hour until people have worked 40 hours. After 40 hours labor costs \$15 per hour. Develop a piecewise function for cost as a function of hours worked: $c = C(t)$. Evaluate this function to find the cost of 35 hours of labor, then 40 hours and 45 hours. It may be easier to do the evaluation first and consider how the calculation was done before trying to define the function. Graph the function.*

Solution to #2:

Things will become a lot more transparent if we look at the hourly RATE of labor as a function $L(t)$ of time t. A very simple function indeed because it stays constant 10 dollars per hour if $0 \leq t \leq 40$; it jumps from \$10 to \$15 per hours at time $t = 40$ and it remains at that value:

$$L(t) \;=\; \begin{cases} 10 & \textit{if}\, 0 \leq t \leq 40, \\ 15 & \textit{if}\, t > 40. \end{cases}$$

Think of what happens if a person works for 60 hours and then you know what to do: This is illustrated in Figure 3: "Problem B1: Cost C(t) with underlying rate L(t)" below.

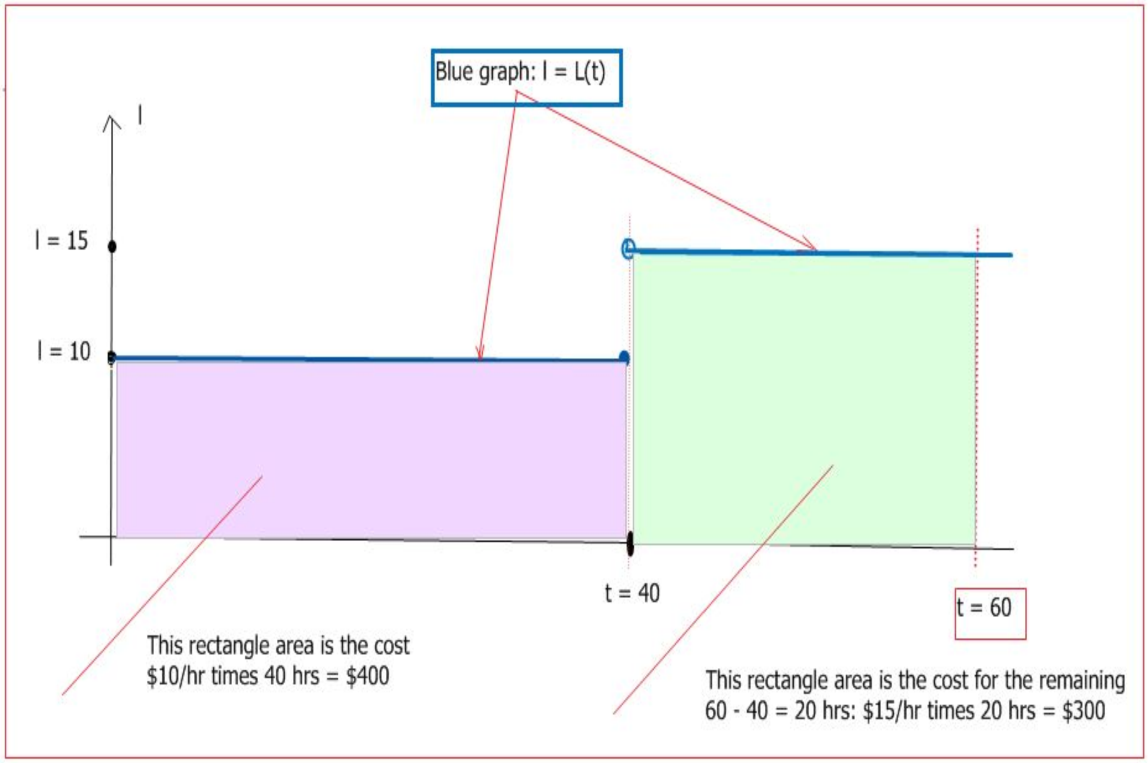


Figure 3: Problem 2: Cost C(t) with underlying rate L(t)

The cost for the first 40 *hours is the area* $40 \times 10 = \$400.00$ *of the left rectangle. To this we must add the cost for the remaining time from* 40 *to* 60 *which is represented by the area* $(60 - 40) \times 15 = \$300.00$ *of the right rectangle.*

It should now be clear how to define the cost function for general t*:*

$$C(t) \;=\; \begin{cases} 10\,t & if\, 0 \leq t \leq 40, \\ 400 + 15(t - 40) & if\, t > 40. \end{cases}$$

Exercise 3: *A product is sold for* $\$7.00$ *each unless it is bought in bulk. If* 100 *items or more of the product are purchased, the cost is* $\$6.50$ *each. Develop a function* $r = R(q)$ *for revenue* r *as a function of items sold* q*. Find the revenue from the sale of* 50 *items,* 100 *items and* 200 *items. Graph the function.*

Solution to #3:

This one is straightforward: We choose symbols that fit the entities they represent:

$$\begin{aligned} q \;&\rightsquigarrow\; \textit{quantity of items sold in a particular sale} \\ r \;&\rightsquigarrow\; \textit{revenue of a particular sale} \\ r = R(q) \;&\rightsquigarrow\; \textit{revenue function dependent on quantity } q \end{aligned}$$

Revenue is $\$7.00q$ *if* $q < 100$ *and* $\$6.50q$ *if* $q \geq 100$*:*

$$R(q) \;=\; \begin{cases} 7q & if\, 0 \leq q < 100, \\ 6.5q & if\, t \geq 100. \end{cases}$$

A graph of this function is shown in Figure 4: "Problem B2: Revenue $r = R(q)$" below.

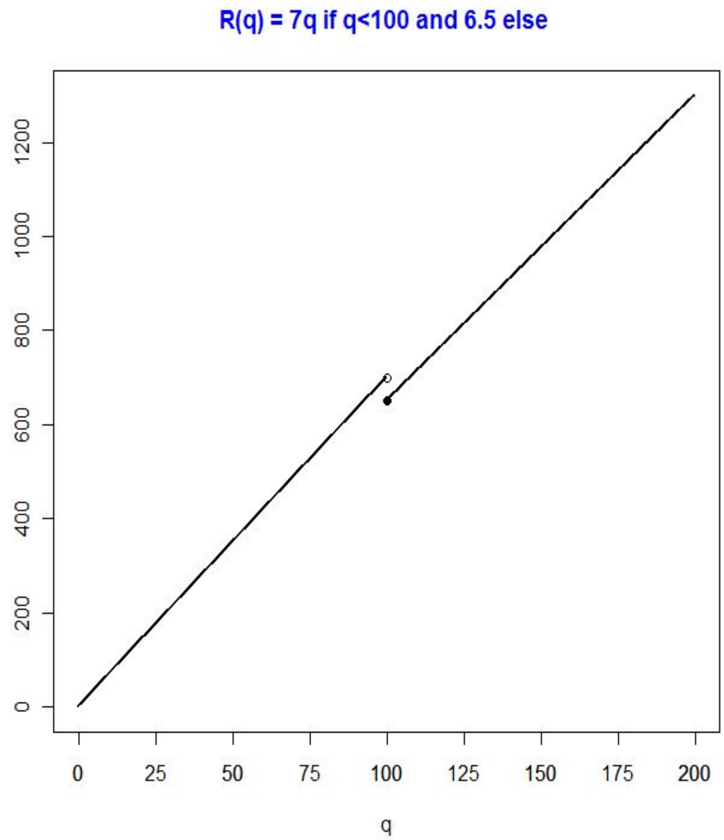


Figure 4: Problem 3: Revenue $r = R(q)$

Points of non-continuity

Exercise 4: *For the following functions examine what happens at points where you would divide by zero and/or (the last two functions) where the function definition changes: Is there a limit? If so, is it finite? If not, what about the one-sided limits?*

$$\textbf{4a}: \frac{5}{6x-4}, \quad \textbf{4b}: \frac{3x+2}{x^2-x-6}, \quad \textbf{4c}: \frac{x^2-2x-35}{x^2-5x-14},$$

$$\textbf{4d}: f(x) = \begin{cases} 1-2x & \text{if } x \le 1, \\ 1/(x-2) & \text{if } 1 < x < 3, \\ x+2 & \text{if } x \ge 3; \end{cases} \qquad \textbf{4e}: f(x) = \begin{cases} x^2+4 & \text{if } x < 2, \\ (x+6)/(x-1) & \text{if } 2 < x \le 4, \\ x-2/3 & \text{if } x > 4. \end{cases}$$

Solution to #4:

We note the following for the first four problems: Each one of them has at least one x-value for which we divide by zero (for C4 this value would be $x = 2$). Of course this means that those x-values are excluded from the domains of those functions but we cannot simply let go: those values can be approached from the left and from the right by x-values that do belong to the domain and we must find out about the one-sided limits and then draw some conclusions about the function behaviour at nearby those points.

Solution to #4a *We blow up for* $6x-4=0$, *i.e., for* $x=2/3$ *because*

$$\lim_{x\to(2/3)-}\frac{5}{6x-4} \;=\; -\infty, \qquad \lim_{x\to(2/3)+}\frac{5}{6x-4} \;=\; \infty.$$

Left and right limit do not coincide, hence $\lim_{x\to 2/3} 5/(6x-4)$ *does not exist. We also note that we have a vertical asymptote at* $x=2/3$.

Solution to #4b

$$\frac{3x+2}{x^2-x-6} \;=\; \frac{3x+2}{(x-3)(x+2)}$$

and we must examine more closely what happens for $x=3$ *and* $x=-2$, *the two points where we would divide by zero. The numerator is not zero for either value and we know right away that the one-sided limits exist and will be either* $-\infty$ *or* ∞. *We plug in values close those points from the left and from the right, e.g.,* $-2.01, -1.99, 2.99, 3.01$, *and we see that in both cases we switch from negative on the left to positive on the right. It follows that*

$$\lim_{x\to(-2)-}\frac{3x+2}{x^2-x-6} \;=\; -\infty, \qquad \lim_{x\to(-2)+}\frac{3x+2}{x^2-x-6} \;=\; \infty,$$
$$\lim_{x\to 3-}\frac{3x+2}{x^2-x-6} \;=\; -\infty, \qquad \lim_{x\to 3+}\frac{3x+2}{x^2-x-6} \;=\; \infty,$$

i.e., right-sided and left-sided limits are different at both $x=-2$ *and* $x=3$ *and the two-sided limit does not exist for either point. Both* $x=-2$, $x=3$ *have vertical asymptotes.*

Solution to #4c

$$\frac{x^2-2x-35}{x^2-5x-14} \;=\; \frac{(x-7)(x+5)}{(x-7)(x+2)} \;=\; \frac{(x+5)}{(x+2)} \quad \textit{if } x\neq -2, x\neq 7.$$

and we must examine more closely what happens for $x=7$ *and* $x=-2$, *the two points where we would divide by zero. For* $x=-2$ *we have the same type of behavior as in a and b: the function blows up in different directions as we approach* -2 *from the left and from the right; we have a vertical asymptote and the two-sided limit does not exist. For* $x=7$ *the situation is different: We can plug* 7 *into the simplified expression* $(x+5)/(x+2)$ *and obtain*

$$\lim_{x\to 7}\frac{x^2-2x-35}{x^2-5x-14} \;=\; 12/9 \;=\; 4/3.$$

It follows that a) the two-sided limit exists (we just computed it!). We could "fix" the function by adding $x=7$ *to its domain and decreeing its value there to be* $12/5$.

Solution to #4d

We note that each of three pieces is well behaved as far as doing something illegal like dividing by zero or taking a negative square root or logarithm is concerned with only one exception: We would divide by zero for $x=2$. *Lets worry about this later and first look at what is happening at the "splicing points"* $x=1$ *and* $x=3$ *where the function definition on the left is different from that on the right. We obtain left-sided and right sided limits at those points simply by plugging the* x*-value into the left side and the right side:*

$$x=1: \quad \lim_{x\to 1-} f(x) \;=\; f(1) = 1-2\cdot 1 \;=\; -1, \quad \lim_{x\to 1+} f(x) \;=\; 1/(1-2) \;=\; -1,$$
$$\textit{i.e., } \lim_{x\to 1-} f(x) \;=\; \lim_{x\to 1+} f(x) \;=\; f(1), \quad \textit{hence } f(x) \textit{ is continuous at } x=1$$
$$x=3: \quad \lim_{x\to 3-} f(x) \;=\; 1/(3-2) \;=\; 1, \quad \lim_{x\to 3+} f(x) \;=\; 3+2 \;=\; f(3) \;=\; 5,$$
$$\textit{i.e., } \lim_{x\to 3-} f(x) \;\neq\; f(3), \quad \textit{hence } f(x) \textit{ is not continuous at } x=3 \textit{ (jump discontinuity!)}$$

Now to the unpleasant part: We note that the "division by zero point" 2 is part of the middle segment $1 < x < 3$ and we must take a closer look. First off, 2 does not belong to the domain D_f and we know that x is not continuous at 2 because continuity is restricted to the points x in the domain. Secondly, for $x = 2$ we have the indeterminate form "$1/0$". This means that $f(x)$ blows up to $\pm\infty$ as x approaches 2 from either side and we have a vertical asymptote. More precisely, if x comes from the left then $x - 2$ is negative, hence $f(x) = 1/(x-2)$ is negative, hence $\lim_{x\to 2-} f(x) = -\infty$. On the other hand, if x comes from the right, then $x - 2$ is positive, hence $f(x) = 1/(x-2)$ is positive, hence $\lim_{x\to 2+} f(x) = \infty$.

Solution to #4e

We note that each of three pieces is well behaved as far as doing something illegal like dividing by zero or taking a negative square root or logarithm is concerned with only one exception: The middle piece where $f(x) = (x+6)/(x-1)$ tells us that we would divide by zero for $x = 1$. But this defintion of $f(x)$ is restricted to $2 < x \leq 4$, a segment to which $x = 1$ does not belong: The denominator will never be zero!
Now to the "splicing points" $x = 2$ and $x = 4$ where the function definition on the left is different from that on the right. We obtain left-sided and right sided limits at those points simply by plugging the x-value into the left side and the right side:

$$x = 2: \quad \lim_{x\to 2-} f(x) = f(2) = 2^2 + 4 = 8, \quad \lim_{x\to 2+} f(x) = (2+6)/(2-1) = 8,$$

$$\text{i.e., } \lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x). \quad \text{It follows that } \lim_{x\to 2} f(x) \text{ exists.}$$

BUT $f(x)$ is not continuous at $x = 2$ because $f(2)$ has not been defined!

$$x = 4: \quad \lim_{x\to 4-} f(x) = (4+6)/(4-1) = 10/3 = f(4), \quad \lim_{x\to 4+} f(x) = 4 - 2/3 = 10/3,$$

$$\text{i.e., } \lim_{x\to 4-} f(x) = f(4) = \lim_{x\to 4+} f(x), \quad \text{hence } f(x) \text{ is continuous at } x = 4.$$

Note that $f(x)$ has a removable discontinuity at $x = 2$: If we add 2 to the domain D_f of f and define $f(2) = 8$ then this "extension" of f is continuous for all real numbers.