Math 220 - Calculus f. Business and Mgmt - Worksheets 10, 11

Solutions for Worksheet 10 - 11 - Limits as x Approaches a

Basic limits

Exercise 1:

1a: $lim_{x\to 3}x^2+2x-5$, *1b:* $lim_{x\to 3}\sqrt{x^2+16}$, *1c:* $lim_{x\to 1}\ln(x)$, *1d:* $lim_{x\to 0}e^x$

Solution to #1:

Each of those functions is continuous at the point a where the limit is taken and you are allowed to compute $lim_{x\to a}f(x) = f(a)$:

1a: $lim_{x\to 3}x^2+2x-5 \ = \ 3^2+2(3)-5 \ = \ 9+6-5 \ = \ \boxed{10}$

1b: $lim_{x\to 3}\sqrt{x^2+16} \ = \ \sqrt{3^2+16} \ = \ \sqrt{25} \ = \ \boxed{5}$

1c: $lim_{x\to 1}\ln(x), \quad = \ \ln(1) \ = \ \boxed{0}$

1d: $lim_{x\to 0}e^x \ = \ e^0 \ = \ \boxed{1}$

Simple Fractions

Exercise 2:

2a: $\lim_{x\to 2}\dfrac{x^3+2x-3}{7}$, *2b:* $\lim_{x\to 3}\dfrac{(x-2)(x-3)}{(x-3)}$, *2c:* $\lim_{x\to 1}\dfrac{3x^5-7x+2}{4x^2+6}$,

2d: $\lim_{x\to -2}\dfrac{3x^3+6x^2}{x+2}$, *2e:* $\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$,

Solution to #2:

2a and 2c: Each of those functions is continuous at the point a where the limit is taken and you are allowed to compute $lim_{x\to a}f(x) = f(a)$.

2a: $lim_{x\to 2}\dfrac{x^3+2x-3}{7} \ = \ \dfrac{2^3+2(2)-3}{7} \ = \ \boxed{9/7}$

2c: $lim_{x\to 1}\dfrac{3x^5-7x+2}{4x^2+6} \ = \ \dfrac{3(1^5)-7(1)+2}{4(1^2)+6} \ = \ \dfrac{-2}{10} \ = \ \boxed{-1/5}$

2b, 2d, 2e: We can cancel the denominator against part of the numerator (for $x \neq a$). The part g(x) that remains is continuous at a and we can compute $lim_{x\to a}g(x) = g(a)$:

2b: $\lim_{x\to 3}\dfrac{(x-2)(x-3)}{(x-3)} \ = \ \lim_{x\to 3}(x-2) \ = \ 3-2 \ = \ \boxed{1}$

2d: $\lim_{x\to -2}\dfrac{3x^2\cdot(x+2)}{x+2} \ = \ \lim_{x\to -2} 3x^2 \ = \ \boxed{12}$

2e: $\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-2} \ = \ \lim_{x\to 4}\dfrac{(\sqrt{x}+2)\sqrt{x}-2}{\sqrt{x}-2} \ = \ \lim_{x\to 4}(\sqrt{x}+2) \ = \ \boxed{4}$

Composite Functions

Exercise 3:

3a: $\lim_{x\to 4} e^{x+2}$, *3b:* $\lim_{x\to 4} \ln(x^2+5x+2)$, *3c:* $\lim_{x\to 8} \ln(x-12)$,

3d: $\lim_{x\to 2} \sqrt[3]{5x^2+x+5}$,

Solution to #3:

3a: $\lim_{x\to 4} e^{x+2} = e^6$.

3b: $\lim_{x\to 4} \ln(x^2+5x+2) = \ln(4^2+5(4)+2) = \ln(38)$.

3c: $\lim_{x\to 8} \ln(x-12)$ *DOES NOT EXIST, since* $\ln(x)$ *is undefined when it has a negative input. If we plug in* 8*, we get* $\ln(8-12) = \ln(-4)$*, which is bad.*

3d: $\lim_{x\to 2} \sqrt[3]{5x^2+x+5} = \sqrt[3]{5(2)^2+2+5} = 3$

Piecewise Functions

Exercise 4:

$$\textbf{\textit{4a}} : f(x) = \begin{cases} 2x+3 & \textit{if } x < 3, \\ x^2 & \textit{if } x \geq 3. \end{cases} \quad \textit{Evaluate limits as } x \textit{ approaches } -5, 3 \textit{ and } 6.$$

$$\textbf{\textit{4b}} : f(x) = \begin{cases} 4x & \textit{if } x < 2, \\ x+3 & \textit{if } x \geq 2. \end{cases} \quad \textit{Evaluate limits as } x \textit{ approaches } 0, 2 \textit{ and } 5.$$

$$\textbf{\textit{4c}} : f(x) = \begin{cases} 3x+1 & \textit{if } x < 1, \\ x+3 & \textit{if } x > 1. \end{cases} \quad \textit{Evaluate limits as } x \textit{ approaches } -3, 1 \textit{ and } 4.$$

Identify points of discontinuity and compare your findings to the answers to the same question on the worksheet for class 9.

Solution to #4a:

$\lim_{x\to -5^-} f(x) = 2(-5)+3 = -7$. *Also,* $\lim_{x\to 6^-} f(x) = 6^2 = 36$

Finally, since 3 *is the place where the function splits, we have to test both directions there. So,* $\lim_{x\to 3^-} f(x) = 2(3)+3) = 9$*. And,* $\lim_{x\to 3^+} f(x) = 3^2 = 9$*. The limits on both sides agree, so we say* $\lim_{x\to 3} f(x) = 9$*, and there are no points of discontinuity.*

Solution to #4b:

$\lim_{x\to 0^-} f(x) = 4(0) = 0$. *Also* $\lim_{x\to 5^-} f(x) = 5+3 = 8$.

Again, we have to test both directions where $f(x)$ *splits. This is at* $x = 2$*. So,* $\lim_{x\to 2^-} f(x) = 4(2) = 8$*, and also* $\lim_{x\to 2^+} f(x) = 2+3 = 5$*. So, the limit doesn't exist at* 2*, and* 2 *is also a point of discontinuity.*

Solution to #4c:

$\lim_{x\to -3^-} f(x) = 3(-3)+1 = -8$. *Also,* $\lim_{x\to 4^-} f(x) = 4+3 = 7$.

We have to test both directions at $x = 1$. $\lim_{x\to 1^-} f(x) = 3(1) + 1 = 4$, and also $\lim_{x\to 1^+} f(x) = 1 + 3 = 4$. Both directions agree, so we say $\lim_{x\to 1} f(x) = 4$. Notice there is indeed a point of discontinuity at $x = 1$, even though the limit exists there. This is because the function is not defined at $x = 1$ (look carefully!). Because it is not defined at $x = 1$ you would have to "pick up your pencil" to draw it.

Fractions of the form non-zero / zero

Exercise 5:

5a: $\displaystyle\lim_{x\to 1^+} \frac{x^2}{x-1}$, *5b:* $\displaystyle\lim_{x\to -2^+} \frac{x^3 - 6x^2 + 2}{x+2}$, *5c:* $\displaystyle\lim_{x\to -3^+} \frac{x^3 + 6x}{x^2 + x - 6}$, *5d:* $\displaystyle\lim_{x\to 2^-} \frac{x^3 + 6x}{x^2 + x - 6}$.

Solutions to #5:

5a: The numerator is going to 1, and the denominator is getting arbitrarily close to 0 as x gets closer and closer to 1. The denominator is always positive, since we are approaching 1 from the right side. So the limit is ∞.

5b: The numerator is going to -30 (just plug in -2 in the numerator). The denominator is getting arbitrarily close to 0 as x gets closer and closer to -2. The denominator is always positive, since we are approaching -2 from the right. So the limit is $-\infty$.

5c: The numerator is going to -45 (just plug in -3 in the numerator). If you factor the denominator, you get $(x + 3)(x - 2)$. Since we are approaching -3 from the right, the $(x - 2)$ factor will always have a negative sign, and the $(x + 3)$ factor will have a positive sign. So the numerator has a negative sign, and the negatives in the numerator and denominator cancel out, and the limit is ∞.

5d: This time the numerator goes to 20. The denominator again factors as $(x + 3)(x - 2)$. Since we are approaching 2 from the left, the $(x + 3)$ factor is always positive, and the $(x - 2)$ factor is always negative. Thus, the denominator is always negative. So the limit must be $-\infty$.

Vertical Asymptotes

Exercise 6:

6a: Give the equation(s) for any vertical asymptotes that exist for the fractions in the section above. Note that the last two problems deal with the same function with different limits)

6b: Find vertical asymptotes for *6b.1:* $\displaystyle\frac{3x^3 + 3x^2}{x^2 + 7x + 6}$, *6b.2:* $\displaystyle\frac{x^2 + 5x + 5}{x^2 - 2x - 8}$,

Solution to #6:

Note that vertical asymptotes occur at values of x where the denominator is zero and the numerator is not zero.

6a: Vertical asymptotes for the functions in section E:

For $\frac{x^2}{x-1}$, there is a V.A. at $x = 1$. (See the note above).

For $\frac{x^3-6x^2+2}{x+2}$, there is a V.A. at $x = -2$.

For $\frac{x^3+6x}{x^2+x-6}$, there is a V.A. at $x = -3$ and $x = 2$.

6b: Vertical asymptotes for the functions in F2-a and F2-b:

For 6b.1: Notice that

$$\frac{3x^3 + 3x^2}{x^2 + 7x + 6} = \frac{(3x^2)(x + 1)}{(x + 1)(x + 6)}$$

Notice the $(x+1)$ factors cancel. So, there is a hole at $x=-1$, and a V.A. at $x=-6$.

For 6b.2: Notice that the numerator does not factor nicely, and the bottom factors as $(x-4)(x+2)$. There are two V.A's, one at $x=-2$ and the other at $x=4$.