Math 220 - Calculus f. Business and Management - Worksheet 12

Solutions for Worksheet 12 - Limits as x approaches infinity

Simple Limits

Exercise 1: *Compute the following limits:*

$$\textbf{\textit{1a}}: \lim_{x\to\infty} 3x^2 - 2x + 4 \qquad \textbf{\textit{1b}}: \lim_{x\to\infty} x^5 - 2x^3 + 8 \qquad \textbf{\textit{1c}}: \lim_{x\to-\infty} x^5 - 2x^3 + 8$$

Solution to #1:

For 1a: $\lim_{x\to\infty} 3x^2 - 2x + 4 = \lim_{x\to\infty} x(3x-2) + 4 = \infty$, *since both* x *and* $(3x-2)$ *go to infinity.*

For 1b: $\lim_{x\to\infty} x^5 - 2x^3 + 8 = \lim_{x\to\infty} x^3(x^2-2) + 8 = \infty$, *since both* x^3 *and* (x^2-2) *go to infinity as* x *goes to infinity.*

For 1c: $\lim_{x\to-\infty} x^5 - 2x^3 + 8 = \lim_{x\to-\infty} x^3(x^2-2) + 8 = -\infty$, *since* x^3 *goes to* $-\infty$ *and* (x^2-2) *goes to* ∞.

Exercise 2: *Compute the following limits:*

$$\textbf{\textit{2a}}: \lim_{x\to\infty} \sqrt{x^3 - 2x^2 - 4} \qquad \textbf{\textit{2b}}: \lim_{x\to-\infty} \sqrt{x^3 - 2x^2 - 4} \qquad \textbf{\textit{2c}}: \lim_{x\to-\infty} e^{x^2-3x+4}$$

Solution to #2:

For 2a: $\lim_{x\to\infty} \sqrt{x^3 - 2x^2 - 4} = \lim_{x\to\infty} \sqrt{x^2(x-2) - 4} = \infty$, *since both* x^2 *and* $(x-2)$ *go to infinity.*

For 2b: $\lim_{x\to-\infty} \sqrt{x^3 - 2x^2 - 4} = \lim_{x\to-\infty} \sqrt{x^2(x-2) - 4}$ *which DOES NOT EXIST, since here* x^2 *goes to infinity, and* $(x-2)$ *goes to* $-\infty$. *You cannot take the square root of a negative number, so the limit doesn't exist.*

For 2c: $\lim_{x\to-\infty} e^{x^2-3x+4} = \infty$, *since the exponent is going to infinity.*

Exercise 3: *Compute the following limits:*

$$\textbf{\textit{3a}}: \lim_{x\to\infty} e^{5-x} \qquad \textbf{\textit{3b}}: \lim_{x\to\infty} \ln(2x+5) \qquad \textbf{\textit{3c}}: \lim_{x\to\infty} \ln(1/x)$$

Solution to #3:

For 3a: $\lim_{x\to\infty} e^{5-x} = 0$, *since the exponent is going to* $-\infty$.

For 3b: $\lim_{x\to\infty} \ln(2x+5) = \infty$.

For 3c: $\lim_{x\to\infty} \ln(\frac{1}{x}) = \lim_{x\to\infty} \ln(x^{-1}) = \lim_{x\to\infty} -\ln(x) = -\infty$.

Rational Functions

Exercise 4: *Compute the following limits:*

$$\textbf{\textit{4a}}: \lim_{x\to\infty} \frac{3x^2+2x-1}{4x^2+2} \qquad \textbf{\textit{4b}}: \lim_{x\to\infty} \frac{2x^5+5x^2-6}{4x^2+2} \qquad \textbf{\textit{4c}}: \lim_{x\to-\infty} \frac{2x^5+5x^2-6}{4x^2+2}$$

Solution to #4:

For each problem, we divide everything by the highest power of x in the denominator.

For 4a:

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{4x^2+2} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}{\frac{4x^2}{x^2}+\frac{2}{x^2}} = \frac{3}{4}$$

For 4b:

$$\lim_{x\to\infty} \frac{2x^5+5x^2-6}{4x^2+2} = \lim_{x\to\infty} \frac{\frac{2x^5}{x^2}+\frac{5x^2}{x^2}-\frac{6}{x^2}}{\frac{4x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{2x^3+5-\frac{6}{x^2}}{4+\frac{2}{x^2}} = \infty$$

For 4c:

$$\lim_{x\to-\infty} \frac{2x^5+5x^2-6}{4x^2+2} = \lim_{x\to-\infty} \frac{\frac{2x^5}{x^2}+\frac{5x^2}{x^2}-\frac{6}{x^2}}{\frac{4x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to-\infty} \frac{2x^3+5-\frac{6}{x^2}}{4+\frac{2}{x^2}} = -\infty$$

Exercise 5: *Compute the following limits:*

$$\textbf{5a}: \lim_{x\to-\infty} \frac{x^2+12}{3x^4-7} \qquad \textbf{5b}: \lim_{x\to\infty} \frac{2x^2+5x+7}{x^3-1} \qquad \textbf{5c}: \lim_{x\to-\infty} \frac{2x^2+5x+7}{x^3-1}$$

Solution to #5:

For each problem, we divide everything by the highest power of x in the denominator.

For 5a:

$$\lim_{x\to-\infty} \frac{x^2+12}{3x^4-7} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^4}+\frac{12}{x^4}}{\frac{3x^4}{x^4}-\frac{7}{x^4}} = \frac{0}{3} = 0$$

For 5b:

$$\lim_{x\to\infty} \frac{2x^2+5x+7}{x^3-1} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^3}+\frac{5x}{x^3}+\frac{7}{x^3}}{\frac{x^3}{x^3}-\frac{1}{x^3}} = \frac{0}{1} = 0$$

For 5c:

$$\lim_{x\to-\infty} \frac{2x^2+5x+7}{x^3-1} = \lim_{x\to-\infty} \frac{\frac{2x^2}{x^3}+\frac{5x}{x^3}+\frac{7}{x^3}}{\frac{x^3}{x^3}-\frac{1}{x^3}} = \frac{0}{1} = 0$$

Other Fractions

Exercise 6: *Compute the following limits.*

$$\textbf{6a}: \lim_{x\to-\infty} \frac{3}{\ln(x^2)} \qquad \textbf{6b}: \lim_{x\to-\infty} \frac{-5}{e^x} \qquad \textbf{6c}: \lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+2x-5}}$$

Solution to #6:

In the following $\to$ is a short for "approaches" or "gets close to": "$x \to -\infty$" means x approaches $-\infty$, "$x \to \pi$" means x approaches π (from both sides), "$x \to 3-$" means x approaches 3 from the left, "$x \to 3+$" means x approaches 3 from the right, etc. As usual, "left side $\rightsquigarrow$ right side" means that what's on the left lets you conclude what's on the right. Example: "$3z+6=15 \rightsquigarrow 3z=9 \rightsquigarrow z=3$"

Solution to 6a:

$$x \to -\infty \rightsquigarrow x^2 \to +\infty \rightsquigarrow \frac{3}{\ln(x^2)} \to 0 \rightsquigarrow \lim_{x\to-\infty} \frac{3}{\ln(x^2)} = 0.$$

Solution to 6b:

$$x \to -\infty \rightsquigarrow e^x \to 0+ \rightsquigarrow \frac{-5}{e^x} \to -\infty \rightsquigarrow \lim_{x\to-\infty} \frac{-5}{e^x} = -\infty.$$

Solution to 6c:

$$x \to -\infty \rightsquigarrow \frac{3x}{\sqrt{x^2+2x-5}} \approx \frac{3x}{\sqrt{x^2}} = \frac{3x}{|x|} \textit{ (absolute value!)} = -3$$
$$\rightsquigarrow \lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+2x-5}} = -3.$$

***Exercise 7**: Compute the following limits.*

$$\textbf{\textit{7a}}: \lim_{x\to\infty} \frac{3x^2}{\sqrt{x^5+4x-5}} \qquad \textbf{\textit{7b}}: \lim_{x\to-\infty} \frac{\sqrt{x^6}}{\sqrt{x^2-4x+5}} \qquad \textbf{\textit{7c}}: \lim_{x\to-\infty} \frac{\sqrt[3]{x^7}}{\sqrt{x^3}}$$

Solution to 7a:

$$x \to \infty \rightsquigarrow \frac{3x^2}{\sqrt{x^5+4x-5}} \approx \frac{3x^2}{\sqrt{x^5}} = 3x^{2-5/2} = \frac{3}{\sqrt{x}} \to 0;$$
$$\textit{hence } \lim_{x\to\infty} \frac{3x^2}{\sqrt{x^5+4x-5}} = \lim_{x\to\infty} \frac{3}{\sqrt{x}} = 0.$$

***Solution to 7b** - makes use again of $\sqrt{x^2} = |x|$:*

$$x \to -\infty \rightsquigarrow \frac{\sqrt{x^6}}{\sqrt{x^2-4x+5}} \approx \frac{\sqrt{x^6}}{\sqrt{x^2}} = \frac{|x|^3}{|x|} = |x|^2 = x^2 \to \infty;$$
$$\textit{hence } \lim_{x\to-\infty} \frac{\sqrt{x^6}}{\sqrt{x^2-4x+5}} = \lim_{x\to-\infty} x^2 = \infty.$$

Solution to 7c:

$$x \to -\infty \rightsquigarrow x^3 \to -\infty \rightsquigarrow x^3 < 0 \rightsquigarrow \sqrt{x^3} \textit{ D.N.E.}; \quad \textit{hence } \lim_{x\to-\infty} \frac{\sqrt[3]{x^7}}{\sqrt{x^3}} \textit{ D.N.E.}.$$

Horizontal Asymptotes

***Exercise 8** Find the horizontal asymptotes for each function on the worksheet.*

We shall limit ourselves to the functions in the section "Other Fractions". We remember the following generalities about horizontal asymptotes: A function $f(x)$ has a horizontal asymptote on the left if $\lim_{x\to-\infty} f(x)$ exists as a (finite) real number x_l and $f(x)$ has a horizontal asymptote on the right if $\lim_{x\to+\infty} f(x)$ exists as a (finite) real number x_r ($\pm\infty$ are not permitted for either x_l or x_r). We have two different horizontal asymptotes if both limits exist and are the same (finite) real number. We have a single horizontal asymptote if only one of the limits is a (finite) real number or if $x_l = x_r$.

Horizontal asymptotes for #4a: $f(x) = \dfrac{3}{\ln(x^2)}$

We have seen above that $\lim_{x\to-\infty} \dfrac{3}{\ln(x^2)} = 0$*, hence the line* $y = 0$ *is a horizontal asymptote on the left.*

$$(-x)^2 = x^2 \;\rightsquigarrow\; f(-x) = f(x) \;\rightsquigarrow\; \lim_{x\to\infty} f(x) \;=\; \lim_{x\to-\infty} f(x).$$

It follows that the line $y = 0$ *also is a horizontal asymptote on the right, i.e., this is the only horizontal asymptote for* $f(x)$*.*

Horizontal asymptotes for #4b: $f(x) = \dfrac{-5}{e^x}$
We have seen above that $f(x)$ *blows up to* $-\infty$ *as* $x \to -\infty$ *and there is no horizontal asymptote on the left. What about* $x \to \infty$*? Then*

$$e^x \to \infty \;\rightsquigarrow\; \frac{-5}{e^x} \to 0 \;\rightsquigarrow\; \lim_{x\to\infty} \frac{-5}{e^x} = 0.$$

It follows that the line $y = 0$ *is a horizontal asymptote on the right.*

Horizontal asymptotes for #4c: $f(x) = \dfrac{3x}{\sqrt{x^2+2x-5}}$

We have seen above that $\lim_{x\to-\infty} \dfrac{3x}{\sqrt{x^2+2x-5}} = -3$*, hence the line* $y = -3$ *is a horizontal asymptote on the left.*
Right hand side:

$$x \to \infty \;\rightsquigarrow\; \frac{3x}{\sqrt{x^2+2x-5}} \;\approx\; \frac{3x}{\sqrt{x^2}} \;=\; \frac{3x}{|x|}\,\textit{(absolute value!)} \;=\; +3$$
$$\rightsquigarrow\; \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+2x-5}} = 3,$$

hence the line $y = 3$ *is a horizontal asymptote on the right. In particular, f(x) has two different horizontal asymptotes.*

Horizontal asymptotes for #5a: $f(x) = \dfrac{3x^2}{\sqrt{x^5+4x-5}}$

We have seen above that $\lim_{x\to\infty} \dfrac{3x^2}{\sqrt{x^5+4x-5}} = 0$*, hence the line* $y = 0$ *is a horizontal asymptote on the right.*
Left hand side:

$$x \to -\infty \;\rightsquigarrow\; x^5+4x-5 \;\approx\; x^5 \to -\infty \;\rightsquigarrow\; x^5+4x-5 < 0 \;\rightsquigarrow\; \sqrt{x^5+4x-5}\ \textit{D.N.E.};$$

hence the domain of $f(x)$ *does not extend all the way to the left and there is no horizontal asymptote on the left side.*

Horizontal asymptotes for #5b: $f(x) = \dfrac{\sqrt{x^6}}{\sqrt{x^2-4x+5}}$

We have seen above that $\lim_{x\to-\infty} \dfrac{\sqrt{x^6}}{\sqrt{x^2-4x+5}} = \infty$*, hence* $f(x)$ *blows up and there is no horizontal asymptote on the left.*
Right hand side:

$$x \to \infty \;\rightsquigarrow\; \frac{\sqrt{x^6}}{\sqrt{x^2-4x+5}} \approx \frac{\sqrt{x^6}}{\sqrt{x^2}} = \frac{|x|^3}{|x|} \;=\; |x|^2 \;=\; x^2 \to \infty;$$
$$\textit{hence } \lim_{x\to\infty} \frac{\sqrt{x^6}}{\sqrt{x^2-4x+5}} \;=\; \lim_{x\to\infty} x^2 \;=\; \infty,$$

same situation as on the left: $f(x)$ blows up and there is no horizontal asymptote on the right either. There are no horizontal asymptotes for $f(x)$.

Horizontal asymptotes for #5c: $f(x) = \dfrac{\sqrt[3]{x^7}}{\sqrt{x^3}}$

We have seen above that $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x^7}}{\sqrt{x^3}}$ D.N.E. and there is no horizontal asymptote on the left.
Right hand side:

$$x \to \infty \ \rightsquigarrow \ \frac{\sqrt[3]{x^7}}{\sqrt{x^3}} = \frac{x^{7/3}}{x^{3/2}} = x^{14/6-9/6} = x^{5/6} \to \infty;$$
$$\textit{hence } \lim_{x\to\infty} \frac{\sqrt[3]{x^7}}{\sqrt{x^3}} = \infty.$$

This is not a finite limit and there is no horizontal asymptote either.

In case you were wondering why $\lim_{x\to\infty} x^{5/6} = \infty$: You remember that for a base $a > 1$ the exponential function $G(x) = a^x$ increases with x. This means that $\sqrt{a} = a^{1/2} < a^{5/6}$ for all $a > 1$. Change of names: replace "a" with "x" and you get $\sqrt{x} < x^{5/6}$ for all $x > 1$ which certainly is the case if $x \to \infty$. But you know that $\sqrt{x} \to \infty$ as $x \to \infty$. This means that the bigger values $x^{5/6}$ also go to infinity.