Math 220 - Calculus f. Business and Management - Worksheet 17

Solutions for Worksheet 17 - Leibniz Notation and Higher Order Derivatives

Find the second derivative of each function

Exercise 1: *Note that you previously computed the first dervatives (worksheet 14).*

$$\textbf{\textit{1a}} : f(x) \,=\, x^3 + 2x - 5\,, \qquad \textbf{\textit{1b}} :\; f(x) \,=\, 6x^4 - 3x^2 + 2x - 7\,, \qquad \textbf{\textit{1c}} :\; f(x) \,=\, \sqrt[4]{x}\,, \qquad \textbf{\textit{1d}} : \quad f(x) \,=\, \frac{1}{x^3}\,.$$

Solution to #1*:*

For 1a: $y = x^2 + 2x - 5$*, so* $\dfrac{dy}{dx} = 3x^2 + 2$ *and* $\dfrac{d^2y}{dx^2} = 6x$.

For 1b: $y = 6x^4 - 3x^2 + 2x - 7$*, so* $\dfrac{dy}{dx} = 24x^3 - 6x + 2$ *and* $\dfrac{d^2y}{dx^2} = 72x^2 - 6$.

For 1c: $y = \sqrt[4]{x} = x^{1/4}$*, so* $\dfrac{dy}{dx} = (1/4)x^{-3/4}$ *and* $\dfrac{d^2y}{dx^2} = (1/4)(-3/4)x^{-7/4} = (-3/16)x^{-7/4}$.

For 1d: $y = \dfrac{1}{x^3} = x^{-3}$*, so* $\dfrac{dy}{dx} = -3x^{-4}$ *and* $\dfrac{d^2y}{dx^2} = 12x^{-5}$.

Exercise 2*: Note that you previously computed the first dervatives (worksheet 14).*

$$\textbf{\textit{2a}} :\; f(x) \,=\, \sqrt{x^5}\,, \qquad \textbf{\textit{2b}} :\; f(x) \,=\, \frac{7}{\sqrt{x}}, \qquad \textbf{\textit{2c}} :\; f(x) \,=\, \sqrt[3]{\frac{5}{x^2}}\,.$$

Solution to #2*:*

For 2a: $y = \sqrt{x^5} = x^{5/2}$*, so* $\dfrac{dy}{dx} = (5/2)x^{3/2}$ *and* $\dfrac{d^2y}{dx^2} = (5/2)(3/2)x^{1/2} = (15/4)x^{1/2}$.

For 2b: $y = \dfrac{7}{\sqrt{x}} = 7x^{-1/2}$*, so*

$\dfrac{dy}{dx} \,=\, (7)(-1/2)x^{-3/2} \,=\, (-7/2)x^{-3/2}$ *and* $\dfrac{d^2y}{dx^2} \,=\, (-7/2)(-3/2)x^{-5/2} \,=\, (21/4)x^{-5/2}$.

For 2c: $y = \sqrt[3]{\dfrac{5}{x^2}} = 5^{1/3}x^{-2/3}$*, so* $\dfrac{dy}{dx} = 5^{1/3}(-2/3)x^{-5/3}$ *and* $\dfrac{d^2y}{dx^2} \,=\, 5^{1/3}(-2/3)(-5/3)x^{-8/3} \,=\, \dfrac{10\sqrt[3]{5}}{9\sqrt[3]{x^8}}$.

Exercise 3: *Note that you previously computed the first dervatives for the first two functions (worksheet 15).*

$$\boldsymbol{3a} : f(x) \;=\; e^x\, x^2, \qquad \boldsymbol{3b} : \; f(x) \;=\; \frac{\sqrt{x}}{5e^x}. \qquad \boldsymbol{3c} : \; f(x) \;=\; \ln(3x^2+2x-5).$$

Solution to #3:

For 3a: $y = e^x x^2$, *so* $\dfrac{dy}{dx} = (e^x)(2x) + (e^x)(x^2) = e^x(2x+x^2)$ *and* $\dfrac{d^2y}{dx^2} = e^x(2+2x) + e^x(2x+x^2)$.

For 3b: $y = \dfrac{\sqrt{x}}{5e^x} = \dfrac{1}{5}x^{1/2}e^{-x}$, *so* $\dfrac{dy}{dx} \;=\; \dfrac{1}{5}(x^{1/2}(-e^{-x}) + (\dfrac{1}{2}x^{-1/2})(e^{-x})) \;=\; \dfrac{1}{5}e^{-x}[\dfrac{1}{2}x^{-1/2} - x^{1/2}]$ *and so*

$$\frac{d^2y}{dx^2} \;=\; \frac{1}{5}[-e^{-x}(\frac{1}{2}x^{1/2} - x^{1/2}) + e^{-x}(-\frac{1}{4}x^{-3/2} - \frac{1}{2}x^{-1/3})]$$

For 3c: $y = \ln(3x^2+2x-5)$, *so* $\dfrac{dy}{dx} = \dfrac{1}{3x^2+2x-5}\cdot(6x+2) = \dfrac{6x+2}{3x^2+2x-5}$ *and so we use the quotient rule to obtain:*

$$\frac{d^2y}{dx^2} = \frac{(6)(3x^2+2x-5) - (6x+2)(6x+2)}{(3x^2+2x-5)^2}$$

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Motion problem

Exercise 4: *An object is sliding on a rail in such a way that its position can be described by this equation:*

$s(t) \;=\; t^3 - 9t^2 + 20t\, m$ *(meters)*

4a : *Where will the object be after* 2 *seconds? How fast will it be moving? What will its acceleration be?*

4b : *When will it be at the origin (position* $= 0$ *)?*

4c : *When will its velocity be* $5m/s$ *(meters/second)?*

4d : *When will its acceleration be* $3\,m/s^2$ *(meters/second*^{2}*)?*

Solution setup: *Before we start attacking the four specific problems we first compute velocity and acceleration because we know that we need them later.*

$$\begin{aligned} v(t) \;&= 3t^2 - 18t + 20\, m/s \\ a(t) \;&= 6t - 18\, m/s^2 \end{aligned}$$

Solution for 4a:

$$s(2) \;= 2^3 - 9\cdot 2^2 + 20\cdot 2 \;=\; 8 - 36 + 40 \;=\; 12m.$$

The object will be 12 *meters to the positive side of the origin.*

$$v(2) \;= 3\cdot 2^2 - 18\cdot 2 + 20 \;=\; 12 - 36 + 20 \;=\; -4\, m/s$$

The object is moving at a speed of 4 *meters in the negative direction.*

$$a(2) \; = 6 \cdot 2 - 18 \, m/s^2 \; = \; 12 - 18 \; = \; -6 \, m/s^2$$

The object is accelerating at a rate of $6m/s^2$ *in the negative direction. Note that if* $v(2)$ *had been of opposite sign (i.e.,* $+4m/s$*), then the object would be decelerating at a rate of* $6m/s^2$*.*

Solution for 4b*: We must solve the equation* $s(t) \; = \; 0$ *for* t*.*

$$\begin{aligned} s(t) \; = \; 0 \;\; &\Rightarrow \; t^3 - 9t^2 + 20t \; = \; 0 \;\; \Rightarrow \;\; t(t^2 - 9t + 20) \; = \; 0 \\ &\Rightarrow \; t(t-5)(t-4) \; = \; 0 \;\; \Rightarrow \;\; t \; = \; 0, 4, 5 \, s \end{aligned}$$

The object will be at the origin three times: 0 *seconds,* 4*, seconds,* 5 *seconds.*

Solution for 4c*: We must solve the equation* $v(t) \; = \; 5$ *for* t*.*

$$\begin{aligned} v(t) \; = \; 5 \;\; &\Rightarrow \; 3t^2 - 18t + 20 \; = \; 5 \;\; \Rightarrow \;\; 3t^2 - 18t + 15 \; = \; 0 \;\; \Rightarrow \;\; t^2 - 6t + 5 \; = \; 0 \\ &\Rightarrow \; (t-5)(t-1) \; = \; 0 \;\; \Rightarrow \;\; t \; = \; 1, 5 \, s \end{aligned}$$

The object will have a velocity of $+5 \, m/s$ *after* 1 *second and after* 5 *seconds.*

Solution for 4d*: We must solve the equation* $a(t) \; = \; 3$ *for* t*.*

$$a(t) \; = \; 3 \;\; \Rightarrow \; 6t - 18 \; = \; 3 \;\; \Rightarrow \;\; 6t \; = \; 21 \;\; \Rightarrow \;\; t \; = \; 7/2 \, s$$

The object will accelerate at a rate of $3 \, m/s^2$ *after* 3.5 *seconds.*