Math 220 - Calculus f. Business and Management - Worksheet 18

Solutions for Worksheet 18 - Rates and the Mean Value Theorem

Average rate of change over an interval

Exercise 1: *Find the average rate of change for each function over the given interval*

$$\begin{aligned}
\textbf{\textit{1a}} &: f(t) \;=\; -3t^2 + 32t + 100 \;\; \textit{from}\; t = 1 \;\; \textit{to}\; t = 4\,, \\
\textbf{\textit{1b}} &: f(x) \;=\; \sqrt{x} \;\; \textit{from}\; x = 1 \;\; \textit{to}\; x = 64\,, \\
\textbf{\textit{1c}} &: f(w) \;=\; e^w \;\; \textit{from}\; w = 2 \;\; \textit{to}\; w = 5 \; \textit{(use a calculator to get an approximate answer)}, \\
\textbf{\textit{1d}} &: f(q) \;=\; \log_3 q \;\; \textit{from}\; q = 1 \;\; \textit{to}\; q = 81\,.
\end{aligned}$$

Solution to 1a:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{180 - 129}{3} = \frac{51}{3} = 17.$$

Solution to 1b:

$$\frac{f(64) - f(1)}{64 - 1} = \frac{8 - 1}{64 - 1} = \frac{7}{63} = \frac{1}{9}.$$

Solution to 1c:

$$\frac{f(5) - f(2)}{5 - 2} = \frac{e^5 - e^2}{3} \approx 47.$$

Solution to 1d:

$$\frac{f(81) - f(1)}{81 - 1} = \frac{4 - 0}{80} = \frac{1}{20}.$$

Instantaneous rate of change

Exercise 2: *For the functions in excercise 1, compute their instantaneous rate of change at the given point:*

$$\begin{aligned}
\textbf{\textit{2a}} &: f(t) \;=\; -3t^2 + 32t + 100 : \quad t = 4\,, \\
\textbf{\textit{2b}} &: f(x) \;=\; \sqrt{x} : \quad x = 25\,, \\
\textbf{\textit{2c}} &: f(w) \;=\; e^w : \quad w = 3 \; \textit{(use a calculator to get an approximate answer)}, \\
\textbf{\textit{2d}} &: f(q) \;=\; \log_3 q : \quad q = 9\,.
\end{aligned}$$

Solution to #2: *Computing an instantaneous rate of change at a point x_0 means computing the derivative and it's value at x_0.*

Step 1: Here are the derivatives:

$$\textbf{\textit{2a:}}\; \frac{dy}{dt} = -6t + 32, \qquad \textbf{\textit{2c:}}\; \frac{dy}{dw} = e^w,$$

$$\textbf{\textit{2b:}}\; \frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \qquad \textbf{\textit{2d:}}\; \frac{dy}{dq} = \frac{1}{q \ln 3}.$$

Step 2: Here are their values for the specific arguments given:

$$\textbf{\textit{2a:}}\ \left.\frac{dy}{dt}\right|_4 = 8, \qquad \textbf{\textit{2c:}}\ \left.\frac{dy}{dw}\right|_3 = e^3 \approx 20,$$
$$\textbf{\textit{2b:}}\ \left.\frac{dy}{dx}\right|_{25} = \frac{1}{10}, \qquad \textbf{\textit{2d:}}\ \left.\frac{dy}{dq}\right|_9 = \frac{1}{9\ln 3} \approx 0.10$$

Mean Value Theorem

Exercise 3: *Given is $f(x) = \sqrt[3]{x}$. Find the point at which the instantaneous rate of change of $f(x)$ equals its average rate of change from x = 1 to x = 8.*

Solution to #3:

Average rate of change from x = 1 to x = 8:

$$\frac{f(8) - f(1)}{8 - 1} = \frac{2 - 1}{8 - 1} = 1/7$$

Instantaneous rate of change at x equals $f'(x)$:

$$f'(x) = \frac{d}{dx}(x^{1/3}) = x^{-2/3}/3 = \frac{1}{3x^{2/3}}$$

We equate the two and solve for x:

$$\frac{1}{3x^{2/3}} = \frac{1}{7} \Rightarrow 7 = 3x^{2/3} \Rightarrow x^{2/3} = \frac{7}{3} \Rightarrow x = \left(\frac{7}{3}\right)^{3/2} \approx 3.56$$

We note that 3.56 is indeed a solution because $1 < 3.56 < 8$.