Math 220 - Calculus f. Business and Management - Worksheet 20

Solutions for Worksheet 20 - Implicit Differentiation

Exercise 1: *Assume y is a function of x. Find dy/dx using implicit differentiation:*

1a : $8y - x^2y^3 \;=\; 6\,,$ ***1b*** : $xy^2 + yx^2 + 2y \;=\; 3x^4 + 5\,,$ ***1c*** : $y \cdot e^x + 4 \;=\; x^{5/2}y^2\,,$
1d : $x^3 \ln(y) \;=\; 6\sqrt{x}\,.$

In each case we differentiate d/dx both sides of the equation and then solve for y'.

Solution for 1a:

$$8y - x^2y^3 \;=\; 6 \;\;\Rightarrow\;\; 8y' \,-\, (2xy^3 + x^2 \cdot 3y^2y') \;=\; 0 \;\;\Rightarrow\;\; (8 \,-\, 3x^2y^2)y' \;=\; 2xy^3$$
$$\Rightarrow\; y' \;=\; \frac{2xy^3}{8 \,-\, 3x^2y^2}$$

Solution for 1b:

$$y^2 \,+\, x(2y)y' \,+\, y'x^2 \,+\, y(2x) \,+\, 2y' \;=\; 12x^3.$$

So,

$$(x^2 \,+\, 2xy \,+\, 2)y' \;=\; 12x^3 \,-\, 2xy \,-\, y^2,$$

or equivalently,

$$y' \;=\; \frac{12x^3 - 2xy - y^2}{x^2 + 2xy + 2}. \;\blacksquare$$

Solution for 1c:

$$y'e^x \,+\, ye^x \;=\; \frac{5}{2}\,x^{3/2}y^2 \,+\, x^{5/2}(2y)y'.$$

So,

$$(e^x \,-\, 2x^{5/2}y)y' \;=\; \frac{5}{2}\,x^{3/2}y^2 \,-\, e^xy,$$

or equivalently,

$$y' \;=\; \frac{(5/2)x^{3/2}y^2 - e^xy}{e^x - 2x^{5/2}y)} \;=\; \frac{5x^{3/2}y^2 - 2e^xy}{2(e^x - 4x^{5/2}y)}. \;\blacksquare$$

Solution for 1d:

$$3x^2\ln(y) + \frac{x^3}{y}\,y' = \frac{3}{\sqrt{x}},$$
$$\frac{x^3}{y}\,y' = 3\left(\frac{1}{\sqrt{x}} - x^2\ln(y)\right),$$

or equivalently,

$$y' \;=\; 3yx^{-7/2} - 3x^{-1}y\ln(y) \;=\; 3y\left(\frac{1}{\sqrt{x^7}} - \frac{\ln(y)}{x}\right).\ \blacksquare$$

***Exercise 2**: The equation*

$$qp \;= 6000 \;-\; p^2$$

represents how price p and demand q for a product are related. Assume demand is a function of price. First find dq/dp using implicit differentiation. Then solve the equation a second time for q and find dq/dp using normal differentiation. Your two answers should be the same.

We write q' instead of dq/dp

***First solution for #2 - implicit differentiation**:*

$$q'p \;+\; q\cdot 1 \;=\; -2p \;\;\Rightarrow\;\; q'p \;=\; -2p \;-\; q \;\;\Rightarrow\;\; q' \;=\; \frac{-\,2p \;-\; q}{p}$$

***Second solution for #2 - - standard differentiation**:*

Since

$$q \;=\; \frac{6000 - p^2}{p},$$

$$q' = \frac{(0-2p)p - (6000 - p^2)\cdot 1}{p^2} \;=\; \frac{-2p^2 - qp}{p^2} \;=\; \frac{-2p - q}{p}.\ \blacksquare$$

***Exercise 3**: Find an equation for the line tangent to each curve at the given point:*

$$\textit{\textbf{3a}}: 16y^2 + 4x^2 \;=\; 100 \quad \textit{at } (3,2)\,, \qquad \textit{\textbf{3b}}:\; y\sqrt{x} \;=\; 3x\sqrt{y} \;-\; 10 \quad \textit{at } (4,1)\,.$$

***Solution for 3a**:*

d/dx on both sides of the equation gives us $32yy' + 8x \;=\; 0$. We could solve this for y' but at this point we are allowed to use that we specifically deal with $x = 3$ and $y = 2$. We plug that into our last equation and obtain

$$32\cdot 2\cdot y' + 8\cdot 3 \;=\; 0 \;\;\Rightarrow\;\; 64y' \;+\; 24 \;=\; 0 \;\;\Rightarrow\;\; y' \;=\; -24/64 \;\;\Rightarrow\;\; \boxed{y' \;=\; -3/8}$$

We have now that the slope m of the tangent line is $m = y' = -3/8$.
Point-slope form hence is $\boxed{y - 2 \;=\; -3/8\,(x-3)}$.

***Solution for 3b**:*

d/dx on both sides of the equation $yx^{1/2} \;=\; 3xy^{1/2} \;-\; 10$ gives us

$$y'\sqrt{x} + y(1/2)x^{-1/2} \;=\; 3\cdot 1\cdot\sqrt{y} \;+\; 3x(1/2)y^{-1/2}y'\,.$$

We could solve this for y' but at this point we are allowed to use that we specifically deal with $x = 4$ and $y = 1$. We plug that into our last equation and obtain

$$y'\sqrt{4} \;+\; 1(1/2)4^{-1/2} \;= 3\cdot 1\cdot\sqrt{1} \;+\; 3\cdot 4(1/2)1^{-1/2}y' \;\;\Rightarrow\;\; 2y' \;+\; (1/2)(1/2) \;=\; 3 + 6y'$$
$$\Rightarrow\;\; -\,3 \;+\; 1/4 \;=\; 4y' \;\;\Rightarrow\;\; -11/4 \;=\; 4y' \;\;\Rightarrow\;\; \boxed{y' \;=\; -11/16}$$

We have now that the slope m of the tangent line is $m = y' = -11/16$.
Point-slope form hence is $\boxed{y - 1 \ = \ -(11/16)\,(x - 4)}$