Math 220 - Calculus f. Business and Management - Worksheets 21 & 22

Solutions for Worksheets 21, 22 - Related Rates

This worksheet contains six related rates problems To solve such a problem you must perform the following two tasks:

Task 1: Read each problem, identify the known and desired rates and quantities. Draw a picture if necessary and write the appropriate equation.

Task 2: Take the derivative of both sides of the equation with respect to time. Solve the new equation for the desired rate.

You can perform the first task for all problems you intend to solve and then perform the second task or you can completely solve a problem and then move on to the next one. Each approach has its advantages: The first one is good if focusing on a single task until it "sticks" works well for you, the second one is good if you need the instant gratification of having the result for a problem to keep on working the problems.

Exercise 1: In this problem the equation is supplied so it is only necessary to identify the known and desired rates. The demand equation for a product shows us that the quantity produced varies with the price according to the equation $q = 1,200/p$. The price is increasing at a rate of $0.06 per month. How fast is the demand for this product changing when the price is $6.00?

Solution for #1:

Since $q = 1200/p$ and p is a function of time $p(t)$, q is a function of time $q(t)$. Price is a linear function of time; so, as a polynomial, it is a differentiable function. So q, being a composition of two differentiable functions, is a differentiable function of time. According to the Chain Rule for Differentiation,

$$\frac{dq}{dt} = \frac{dq}{dp} \cdot \frac{dp}{dt} = -\frac{1200}{p^2} \cdot \frac{dp}{dt}.$$

We also know that $dp/dt = 0.06$ at any time t. At the specific time, $t_\circ$, that the price is $6, the rate in items per month that the demand is changing is

$$\left.\frac{dq}{dt}\right|_{t_\circ} = \left(-\frac{1200}{p^2}\right)(0.06) = -2.$$

Exercise 2: The demand function for a product is $q = 1,000 - 0.2\sqrt{p}$. The price is increasing at a rate of $0.10 per week. How fast is the revenue changing when the price of one unit is $36.00? Hint: First find the revenue as a function of price.

Solution for #2:

Since $q = 1000 - 0.2\sqrt{p}$, the revenue is

$$r = qp = (1000 - 0.2\sqrt{p})p = 1000p - 0.2p^{3/2} = 1000p - \frac{2}{10}p^{3/2}.$$

Since the price p is a function of time $p(t)$, q is a function of time $q(t)$. Price is a linear function of time; so, as a polynomial, it is a differentiable function of time. So r as a difference of two differentiable functions of time, is another differentiable function of time. According to the Chain Rule for Differentiation,

$$\frac{dr}{dt} = \frac{dr}{dp} \cdot \frac{dp}{dt} = \left(1000 - \frac{2}{10}\left(\frac{3}{2}\sqrt{p}\right)\right)\frac{dp}{dt} = 1000 \cdot \frac{dp}{dt} - \left(\frac{3}{10}\sqrt{p}\right)\frac{dp}{dt}.$$

We have $dp/dt = 0.1$ for all t and we are interested what happens to dr/dt at the time $t_\circ$ where the price is $36, i.e., $p = 36$. We plug $dp/dt = 0.1$ and $p = 36$ into the last equaton and get

$$\left.\frac{dr}{dt}\right|_{t_\circ} = 1000(0.1) - \left(\frac{3}{10}\sqrt{36}\right)(0.1) = 100 - \frac{18}{100} = 99.82 \text{ dollars per week.}$$

We finally note that the amount is positive, i.e., revenues are increasing and not falling at that rate.

Exercise 3: A pebble is dropped into a pond. The area of the circle enclosed by the outermost ripple is increasing at a rate of $20cm^2$ per second. How fast is the radius of that ripple increasing when the area inside the circle is 64π?

First solution for #3, without implicit differentiation:

Write r for the radius and and A for the area enclosed by the ripple. Then $A = \pi r^2$.

$$\text{Equivalently,} \quad r = r(A) = \sqrt{A/\pi} = A^{1/2}/\sqrt{\pi} \Rightarrow \frac{dr}{dA} = \frac{1}{\sqrt{\pi} \cdot 2\sqrt{A}} = \frac{1}{2\sqrt{\pi A}}$$

Area is a linear function of time; so, it is a differentiable function and r, being a composition of two differentiable functions of time, is another differentiable function of time. According to the Chain Rule for Differentiation,

$$\frac{dr}{dt} = \frac{dr}{dA}\frac{dA}{dt} = \frac{1}{2\sqrt{\pi A}} \cdot \frac{dA}{dt}.$$

We have $dA/dt = 20$ for all t and we are interested what happens to dr/dt at the time $t_\circ$ where $A = 64\pi$. We plug $dA/dt = 20$ and $A = 64\pi$ into the last equaton and get

$$\left.\frac{dr}{dt}\right|_{t_\circ} = \frac{1}{2\sqrt{\pi(64\pi)}}(20) = \frac{20}{2\sqrt{64\pi^2}} = \frac{20}{2\cdot 8\pi} = \frac{5}{4\pi}. \text{ cm per second.}$$

Second solution for #3, with implicit differentiation:

Write r for the radius and and A for the area enclosed by the ripple. Then $A = \pi r^2$. We are interested in dr/dt. The cookbook tells us to differentiate $A = \pi r^2$ (implicitly) with respect to t.

$$(\star) \quad \frac{dA}{dt} = \pi\frac{d}{dt}\left(r(t)\right)^2 = \pi \cdot (2r(t))r'(t)$$

We were not asked to solve for the time $t_\circ$ where $A = 64\pi$ and we don't have to do so to make use of the info that

$$A = 64\pi \,, \text{i.e.,} \quad r^2\pi = 64\pi \,, \text{i.e.,} \quad r = \pm 8 \,, \text{i.e.,} \quad r = 8$$

because there is no such thing as a negative radius. We remember that $dA/dt = 20$ for all times t. We plug that and $r = 8$ into equation $(\star)$ and we get

$$(\star\star)20 = \pi \cdot 2 \cdot 8 r'(t_\circ) \,, \text{i.e.,} \quad r'(t_\circ) = \frac{20}{16\pi} \,, \text{i.e.,} \quad r'(t_\circ) = \frac{5}{4\pi}$$

The radius of the outermost ripple changes at a rate of plus $5/(5\pi)$.

Exercise 4: A child is standing beside a straight river that is $100m$ wide. A boat is moving down the center of the river at a speed of $6m/sec$. How fast is the distance between the child and the boat changing when the boat is 75 meters from the child?

Solution for #4:

We use the following notation:

$\quad P:\quad$ the point that is halfway across the river and directly in front of the child,
$\quad b:\quad$ distance in meters between the boat and P,
$\quad c:\quad$ distance in meters between the boat and the child.

We note that b and c actually are functions $b(t), c(t)$ of time t. According to the Pythagorean Theorem,

$$50^2 + b^2 = c^2.$$

b is a linear function of time; so, b and b^2 are differentiable functions. c^2, being the sum of a differentiable function of time and a constant, is another differentiable function of time. Implicit differentiation gives

$$0 \; + \; 2b\frac{db}{dt} \;=\; 2c\frac{dc}{dt}, \;\; \text{i.e.,} \;\; bb' \;=\; cc'$$

We know that $b' = 6$ at all times. At the time, $t_\circ$, that the boat is 75 meters past point P, Pythagoras gives us

$$50^2 + b^2 \;=\; 75^2 \;=\; (50+25)^2 \;=\; 50^2 + 2\cdot 50 \cdot 25 + 25^2$$
$$\Rightarrow \;\; b^2 \;=\; 75^2 \;=\; 4\cdot 25^2 + 25^2 \;=\; 25^2(4+1) \;\; \Rightarrow \;\; b \;=\; \pm 25\sqrt{5}.$$

No negative distance, hence $b = 25\sqrt{5}$ meters. Let's collect all we have got so far for $t_\circ$:

$$b(t_\circ)b'(t_\circ) \;=\; c(t_\circ)c'(t_\circ), \qquad b'(t_\circ) \;=\; 6, \qquad b(t_\circ) \;=\; 25\sqrt{5}, \qquad c(t_\circ) \;=\; 75$$

We plug items 2-4 into the first equation and get $150\sqrt{5} = 75b'(t_\circ)$, i.e., $b'(t_\circ) = 2\sqrt{5}$. Here is the big catch: The problem did not specify whether the boat is moving towards the child or away from the child. In either case the solution is the same except for the sign: If the boat moves away then the distance increases, so the rate of change is positive and we get $b'(t_\circ) = 2\sqrt{5}$. If the boat moves towards the child then the distance decreases, so the rate of change is negative and we get $b'(t_\circ) = -2\sqrt{5}$.

Exercise 5: A cylindrical container with a diameter of 3 meters is being filled with water. The water is flowing into the container at a rate of $5m^3/sec$. How fast is the water moving up the side of the container?

Solution for #5:

We use the following notation:

$\quad V:\quad$ the amount (volume) of water in the cylinder (cubic meters),
$\quad d:\quad$ diameter of the tank: $d = 3$ meters,
$\quad r:\quad$ radius of the tank: $r = d/2 = 3/2$ meters,
$\quad h:\quad$ height of water in the tank (NOT the height of the tank itself) in meters.

The formula "Volume of water = area of base $\times$ height of water" gives the formula that ties everything together:

$$V \;=\; \pi r^2 h \;=\; \frac{9\pi}{4}\cdot h$$

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We note that a) "water is flowing into the container at a rate of $5m^3/sec$" translates into $\dfrac{dV}{dt} = 5$ and b) the item we want to compute is dh/dt, the rate at which the height of the water goes up with respect to time.

$$\text{Differentiation } d/dt: \quad \frac{dV}{dt} = \pi r^2 h = \frac{9\pi}{4}\cdot\frac{dh}{dt}, \quad \Rightarrow \quad 5 = \frac{9\pi}{4}\cdot\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{5\cdot 4}{9\pi} = \frac{20}{9\pi}$$

The rate at which the height of the water goes up with respect to time is $h'(t) = 20/(9\pi)$ meters per second.

Exercise 6: Andrea and Brad start walking away from home at the same time. Andrea is walking east at $5km/h$ (km per hour). Bob is walking south at $6km/h$. How fast are they moving away from each other after half an hour?

Solution for #6:

We use the following notation:

$a = a(t):$ Ann's distance from home,
$b = b(t):$ Brad's distance from home,
$c = c(t):$ distance between Ann and Brad.

Pythagoras' Theorem

$$c^2 = a^2 + b^2 \qquad (\star)$$

puts everything in relation to each other. We note that our task is to compute $dc/dt(1/2)$.

The diagram below shows what happened after $t = 1$ hour: Andrea is 5 km away from home, Brad is 6 km away from home, hence $a(t) = a(1) = 5$, $b(t) = b(1) = 6$ and $c(t) = \sqrt{a(t)^2 + b(t)^2}$ (Pythagoras), i.e., $c(1) = \sqrt{25 + 36} = \sqrt{61}$.

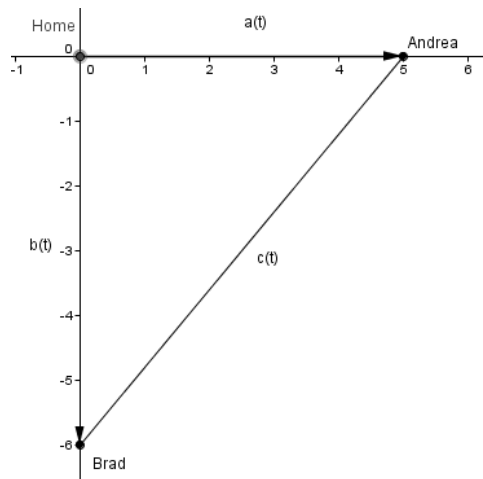

Figure 1: Diagram for problem 6

We differentiate both sides of above formula $(\star)$ (implicitly) with respect to time t:

$$2c(t) \cdot c'(t) = 2a(t) \cdot a'(t) + 2b(t) \cdot b'(t) \qquad (\star\star)$$

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We know how fast Andrea and Bob are walking and this translates into $a'(t) = 5$ m/sec, $b'(t) = 6$ m/sec. Plug that into formula ($\star\star$) and divide everything by 2:

$$c(t) \cdot c'(t) = 5a(t) + 6b(t)$$

That's true for all times t, especially also for our time of interest, $t_\circ = 1/2$, at which $a(1/2) = 5/2$ and $b(1/2) = 3$:

$$c(1/2) \cdot c'(1/2) = 5 \cdot \frac{5}{2} + 6 \cdot 3 = \frac{25 + 36}{2} = \frac{61}{2} \quad \Rightarrow \quad c'(1/2) = \frac{61}{2} \Big/ c(1/2) \qquad (\star\star\star)$$

We can compute $c(1/2)$ from Pythagoras' formula ($\star$):

$$c(1/2)^2 = a(1/2)^2 + b(1/2)^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{6}{2}\right)^2 = \frac{25 + 36}{4} = \frac{61}{4} \quad \Rightarrow \quad c(1/2) = \frac{\sqrt{61}}{2}$$

Plug that last expression for $c(1/2)$ into formula ($\star\star\star$):

$$c'(1/2) = \frac{61}{2} \Big/ \frac{\sqrt{61}}{2} = \frac{61}{2} \cdot \frac{2}{\sqrt{61}} = \sqrt{61}$$