Math 220 - Calculus f. Business and Management - Worksheet 27

Maxima and Minima

Before we start, note the following fine point on relative and absolute extrema. It was written by Michael Fochler when he used a wrong definition for relative extrema in the class room.

Fall 2014 Correction concerning Extremal points

The following applies to the Fall 2014 semester.

I taught the following in both sections 2 and 5 which simply is not true:

*A local maximum a is a point in D_f with the following property: You can find a (very small) positive number $h > 0$ such that the following is true for all numbers x which belong to **BOTH** D_f **AND** the interval $\{x | a-h < x < a+h\} = (a-h, a+h)$: $f(x)$ cannot exceed $f(a)$, i.e., $f(x) \leq f(a)$ for all such x.*

You get from there my definition of a local minimum if you replace "cannot exceed" with "cannot drop below" and "$f(x) \leq f(a)$" with "$f(x) \geq f(a)$".

Here is the correct definition, taken from the top of p.125 of the Brewster/Geoghegan lecture notes:

We say f has a local maximum at $x = a$ if there is a number $h > 0$ such that $f(x) < f(a)$, whenever the distance in $\mathbb{R}$ from x to a is less than h, i.e., when x lies in the open interval $(a-h, a+h)$.

*So what's the big deal? Here it is: The lecture notes' formulation is much more demanding. It implies that I must be able to "surround" a by an entire interval $(a-h, a+h)$ **ALL** points of which belong to the domain D_f.*

So what does my invalid formulation allow that the one from the lecture notes does not allow? The answer: points at the "boundary" of the domain. You cannot make such points a the mid-point of an interval $(a-h, a+h)$ that entirely belongs to D_f, regardless how small a number $h > 0$ you choose.

Example: Look at the function $f(x) = -\sqrt{x}$ on its natural domain $[0, \infty)$ and choose $a = 0$. In my definition 0 would have been a local maximun: Pick $h = 1$. Then the collection of points that belong both to D_f and to $(-1, 1)$ simply is $[0, 1)$ and for any x in $[0, 1)$ you get $f(0) = 0 = -\sqrt{0}$ is at least as big as the negative number $f(x) = -\sqrt{x}$. But 0 is not a local max in the sense of Brewster/Geoghegan because no $h > 0$ can be found such that the interval $(0-h, 0+h) = (-h, h)$ entirely belongs to D_f.

*Are you ready for more? Here it comes: My definition of **ABSOLUTE** max and min is exactly the same as that of Brewster/Geoghegan (see p. 180 at the start of ch.22):*

The absolute maximum of f on a given interval I is the M if (i) there is some a in I such that $f(a) = M$ and (ii) $f(x) \leq M$ for every x in I.

*Note that it has not been demanded that I be an **OPEN** interval (one which does not contain its endpoints). In our example $f(x) = -\sqrt{x}$ we may choose $I = [0, \infty)$. You can see that $M = 0$ is an absolute maximum for f because (i) 0 belongs to I and $f(0) = M$ and (ii) $f(x) \leq 0$ for all $x \geq 0$, i.e., for all x in I.*

You must from now on remember the following:

*It is possible that a function $f(x)$ has a global maximum (or minimum) at a point a but it does **NOT** have a local maximum (or minimum) at that point.*

In the first problem of quiz 5 I gave you the following picture:

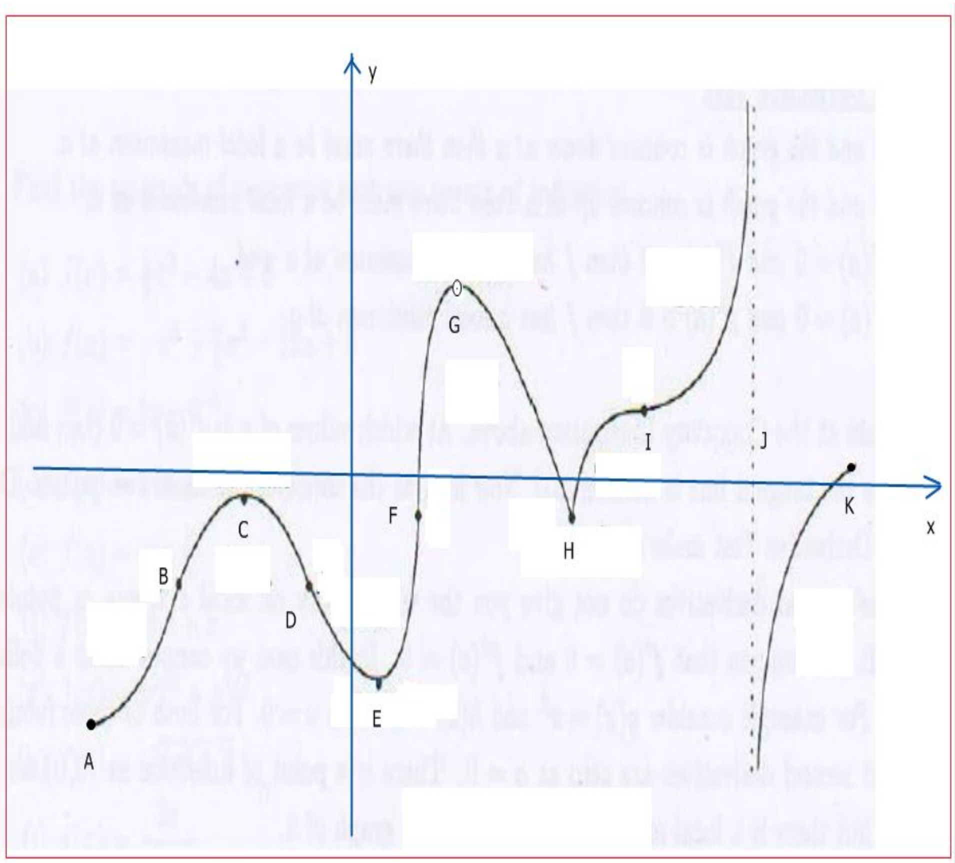

Figure 1: Problem 1: Classify special points

According to my old definition, f would have attained a local minimum at the boundary point A and a local maximum at the boundary point K. I hope you now understand why that is not true.

A final remark: It's not like I said "this" and the Brewster/Geoghegan text said "that". Everyone says what the lecture notes say. I was wrong, plain and simple, and you must learn the correct formulation.

Specifically for the mid-term: you will not be asked to examine boundary points like A abd K in the picture above but I shall not guarantee any such thing for subsequent tests and quizzes, including the final exam. Be sure to unlearn my old teachings and study the correct definitions!

– Michael Fochler –

Keep in mind what was written above when examining the boundary points of the intervals in the following exercises.

***Exercise 1**: Find the local and absolute maxima and minima (extrema) of the following function on each on of the given interval(s). You may use a calculator to find values for $f(x)$. Do not graph the function.*

$$f(x) = 2x^3 - 6x^2 - 48x + 7 \qquad \textit{on the intervals} \quad [-3,6], [-1,5], [-6,8] \textit{ and } (-6,8].$$

Solution to exercise 1:

We compute first and second derivatives to find out about local max/min and the function values for those and all other critical points (the endpoints of the intervals).

$$f'(x) = 6x^2 - 12x - 48, \qquad f''(x) = 12x - 12.$$

$f'(x) = 0$: *same as* $x^2 - 2x - 8 = 0$. *We factor:* $(x-4)(x+2) = 0$, *i.e.,* $x = -2$, $x = 4$.

$$\begin{aligned} f''(-2) &= -24 - 12 < 0 \ \rightsquigarrow \ \textit{local max}, \\ f''(4) &= 48 - 12 > 0 \ \rightsquigarrow \ \textit{local min}. \end{aligned}$$

-2 *and* 4 *are the only points with a horizontal tangent and it follows that* $f(x)$ *is increasing on* $(-\infty, -2)$*, decreasing on* $(-2, 4)$ *and again increasing on* $(4, \infty)$.

Now to the function values:

x	-6	-3	-2	-1	4	5	6	8
f(x)	-353	43	63	47	-153	-133	-65	263

Figure 2: Function values for $f(x) = 2x^3 - 6x^2 - 48x + 7$

You graph those function points and everything now is obvious. Reread the letter above if you do not understand why, e.g., $f(x)$ *does **not** have a local min for* $x = 3$ *if* $D_f = [-3, 6]$.

$$\begin{aligned} [-3,6]: &\quad \textit{abs max f.} \ -2, \ \textit{abs min f.} \ 4; \ \textit{NOT a local max f.} \ 6, \ \textit{NOT a local min f.} -3 \\ [-1,5]: &\quad \textit{abs max f.} \ -1, \ \textit{abs min f.} \ 4; \ \textit{NOT a local max f.} \ 5. \\ [-6,8]: &\quad \textit{abs min f.} -6, \ \textit{local max f.} \ -2, \ \textit{local min f.} \ 4, \ \textit{abs max f.} \ 8. \\ (-6,8]: &\quad \textit{local max f.} \ -2, \ \textit{local min f.} \ 4, \ \textit{abs max f.} \ 8, \textit{abs min D.N.E.} \end{aligned}$$

***Exercise 2**: Find the absolute maxima and minima (extrema) of the following function on the given interval(s). You may use a calculator to find values for $f(x)$. Do not graph the functions.*

$f(x) = \dfrac{4x}{x^2+9}$ *on* $[0,5]$, $[0,8)$ *and* $[1,8)$.

Solution to exercise 2:

We note that $f(x)$ has no vertical asymptotes because the denominator is ≥ 9 for all x.

We compute first and second derivatives to find out about local max/min.

$$f'(x) \;=\; \frac{4(x^2+9) - 4x \cdot 2x}{(x^2+9)^2}$$

We have $f'(x) = 0$ same as numerator $4x^2 + 36 - 8x^2 \;=\; -4x^2 + 36 = 0$, i.e., $x = \pm 3$.

$$f''(x) \;=\; \frac{-\,8x(x^2+9)^2 \;-\; (-4x^2+36) \cdot 2 \cdot 2x(x^2+9)}{(x^2+9)^4}.$$

We note that the factor $(-4x^2 + 36)$ is zero for $x = \pm 3$. Hence

$$\begin{aligned} f''(-3) \;&=\; \frac{24((-3)^2+9)^2}{((-3)^2+9)^4} = \frac{24}{((-3)^2+9)^2} \;>\; 0; \\ f''(3) \;&=\; \frac{-\,24(3^2+9)^2}{(3^2+9)^4} = \frac{24}{(3^2+9)^2} \;<\; 0. \end{aligned}$$

and it follows that we have a local min at -3 and a local max at 3. As the denominator is positive

The function values for the endpoints of the intervals also have to be computed so they can be compared to $f(\pm 3)$. I'll leave that to the class as an exercise.