Math 220 - Calculus f. Business and Management - Worksheet 28

Solutions for Worksheet 28 - Optimization of Business Functions

Exercise 1:

A manufacturer can make a profit of $P(z)$ (in hundreds of dollars) from the sale of z thousand items according to the formula: $P(z) = -z^3 + 9z$. Find the number of items that should be sold to maximize profit. Find out what profit will be made at that point. You may use your calculator to find an approximate answer.

Solution to 1***:*** *Goal is to maximize the profit $P(z) = -z^3 + 9z$ [$100\$/1,000$ items].*

$$\text{Derivative } P'(z) \;=\; -3z^2 + 9 = 0 \;\rightsquigarrow z^2 - 3 = 0 \;\rightsquigarrow\; z \;=\; \pm\sqrt{3}$$

$-\sqrt{3}$ is not in the domain $D_p = [0, \infty)$, hence the only critical point is $\sqrt{3}$.

Second derivative: $P''(z) = -6z$. Hence $P''(z_{\max}) = P''(\sqrt{3}) = -6\sqrt{3} < 0$ and we have a local maximum.

Is it global? We have $P(0) = 0$, $P(\sqrt{3}) = -(\sqrt{3})^2\sqrt{3} + 9\sqrt{3} = 6\sqrt{3}$ and
$\lim_{z\to\infty} P(z) \;=\; \lim_{z\to\infty} P(-z^3) \;=\; -\infty$.

*We found out that a) there is only one point ($z_{\max} = \sqrt{3}$) with a horizontal tangent, b) $P(\sqrt{3}) > P(0) = 0$ and c) $\lim_{z\to\infty} P(z) \;=\; -\infty$. It follows that the profit function $P(z)$ attains a **global** maximum at $z = \sqrt{3}$.*

Exercise 2:

The cost to manufacture x units of a product is $15000 + 40x + 0.02x^2$. The revenue from the sale of these products is $100x - .01x^2$. Find the number of products that will maximize profit.

Solution to 2***:***

$$\begin{aligned}
\text{Cost } C(x) &= 15000 + 40x + 0.02x^2,\\
\text{Revenue } R(x) &= 100x - .01x^2,\\
\text{Profit } P(x) &= R(x) - C(x) \;=\; (-\frac{1}{100} - \frac{2}{100})x^2 + 60x - 15,000 \;=\; -\frac{3}{100}x^2 + 60x - 15,000\\
\text{Derivative } P'(x) &= -\frac{3}{50}x + 60 = 0 \;\rightsquigarrow\; \frac{3}{50}x \;=\; 60 \;\rightsquigarrow\; x \;=\; 1,000\\
2^{nd}\text{derivative } P''(x) &= -\frac{3}{50} < 0 \;\rightsquigarrow\; \text{local maximum at } x = 1,000
\end{aligned}$$

We note that this maximum is global because the function $P(x) = -(3/100)x^2 + 60x - 15,000$ is a parabola symmetric about a vertical axis which points downward (the coefficient $-(3/100)$ of x^2 is negative).

Exercise 3:

A manufacturer makes a product that costs \$12 to produce. He estimates that the demand will be $50 - x$ units when the price is x dollars. Find the cost, revenue and profit equations (as functions of price). Then find the price that maximizes the profit.

Solution to* 3*: *The problem does not say so but we can assume that* \$12.00 *is the cost per unit.*

$$
\begin{aligned}
\textit{Demand } Q(x) \ &= 50 - x, \\
\textit{Cost } C(x) \ &= 12Q(x) \ = \ 12(50-x) \ = \ -12x + 600, \\
\textit{Revenue } R(x) \ &= Q(x)\cdot x \ = \ (50-x)x \ = \ -x^2 + 50x, \\
\textit{Profit } P(x) \ &= R(x) - C(x) \ = \ -x^2 + 62x - 600 \\
\textit{Derivative } P'(x) \ &= \ -2x + 62 \ = \ 0 \ \rightsquigarrow \ x \ = \ 31 \\
2^{nd}\textit{derivative } P''(x) \ &= \ -2 < 0 \ \rightsquigarrow \ \textit{local maximum at } x = 31
\end{aligned}
$$

The profit for the price point of \$31.00*:*

$$
\begin{aligned}
P(31) \ &= \ -31^2 + (2\cdot 31)31 - 600 \ = \ 31^2 - 600 \\
&= (30+1)^2 - 600 \ = \ (900 + 62 + 1) - 600 \ = \ \$363.00
\end{aligned}
$$

This value far exceeds the one for the other critical point, $x = 0$*:* $P(0) = -600$*. It follows that the maximum profit occurs at a price of* \$31.00*.*

Exercise 4*:*

The cost to build q *items is* $6000 + 5q + 0.01q^2$*. In order to sell* q *items, the price will need to be* $p(q) = 20 - q/4$*. Find the quantity that will maximize profits.*

Solution to* 4*:
Notation: We shall write $F(q)$ *rather than* $P(q)$ *for the profit because the letter* P *will denote price.*

$$
\begin{aligned}
\textit{Cost } C(q) \ &= \frac{q^2}{100} + 5q + 6,000 \\
\textit{Price } p(q) \ &= 20 - \frac{q}{4} \\
\textit{Revenue } R(q) \ &= p(q)\cdot q \ = \ (20 - \frac{q}{4})q \ = \ -\frac{q^2}{4} + 20q \\
\textit{Profit } F(q) \ &= R(q) - C(q) \ = \ \big(-\frac{1}{4} - \frac{1}{100}\big)q^2 + (20-5)q - 6,000 \\
&= \ -\frac{26}{100}q^2 + 15q - 6,000 \ = \ -\frac{13}{50}q^2 + 15q - 6,000 \\
\textit{Derivative } F'(q) \ &= \ -\frac{13}{25}q + 15 \ = \ 0 \ \rightsquigarrow \ \frac{13}{25}q \ = \ 15 \ \rightsquigarrow \ q = \frac{15\cdot 25}{13} \\
2^{nd}\textit{derivative } P''(x) \ &= \ -\frac{13}{25} < 0 \ \rightsquigarrow \ \textit{local maximum at } q = \frac{15\cdot 25}{13}
\end{aligned}
$$

The optimal quantity is $q_{\max} = \dfrac{15\cdot 25}{13} \approx 28.85$ *units. To completely finish the problem you should compare* $F(28)$ *and* $F(29)$ *and see which is bigger as it seems unlikely that fractions of a unit can be produced and sold, except for spare parts.*