Math 220 - Calculus f. Business and Management - Worksheet 30

Solutions for Worksheet 30 - Optimization Word Problems

Exercise 1: *A movie theater has a seating capacity of 525. With the ticket price set at \$10, average attendance at a movie has been 375 persons. Management has decided to lower admission prices to boost attendance. A market survey indicates that for each two dollars the price of a ticket is lowered, average attendance will increase by 100.*

a) *Write a function, $q = Q(p)$ to express the the quantity q of tickets sold as a function of the price p.*

b) *What is the range of possible values for the ticket price (consider that the theater can hold no more than 525 people)?*

c) *Use $Q(p)$ to write the revenue function $r = R(p)$.*

d) *To the nearest penny, what ticket price will result in maximum revenue? (Justify your answer)*

Solution to 1***:***

Solution to $1a$***:***

"... for each two dollars the price of a ticket is lowered, average attendance will increase by 100" is mathematically expressed as follows: Let $q = Q(p)$ be the number of tickets sold at price p. If you decrease the ticket price p to $p - 2$ then the quantity $Q(p-2)$ of attendees for that reduced price is 100 more than the quantity $Q(p)$. To write this as a formula: $Q(p-2) = Q(p) + 100$, so $Q(p) - Q(p-2) = -100$. From there we get the difference quotient:

$$\Delta_q/\Delta_p \ = \frac{Q(p) - Q(p-2)}{p - (p-2)} \ = \ \frac{-\,100}{2} \ = \ -50$$

That equation is true for all reasonable ticket prices p: those small enough that attendance $Q(p)$ does not drop off to zero and high enough that we do not sell more tickets $Q(p)$ than the seating capacity of 525.

For those p the slope of $Q(p)$ is constant -50. Hence $Q(p)$ is a straight line with $m = -50$ which goes through the point $(p, q) = (10, 375)$. We get the equation of $Q(p)$ in point-slope form as

$$q - 375 \ = (-50)(p - 10) \ \ \rightsquigarrow \ \ q - 375 \ = \ -50p + 500 \ \ \rightsquigarrow \ \ q \ = \ \boxed{Q(p) \ = \ 875 - 50p}$$

Solution to $1b$***:***

Now that we know the equation for $Q(p)$ we get the domain D_q from the conditions $q, p \geq 0$ and $q \leq 525$.

$$\begin{aligned} q \ \leq \ 525 \ &\rightsquigarrow \ 875 - 50p \ \leq \ 525 \ \rightsquigarrow \ 350 \ \leq \ 50p \ \rightsquigarrow \ p \ \geq \ \$7.00 \\ q \ \geq \ 0 \ &\rightsquigarrow \ 875 - 50p \ \geq \ 0 \ \rightsquigarrow \ 875 \ \geq \ 50p \ \rightsquigarrow \ 100p \ \leq \ 1,750 \ \rightsquigarrow \ p \ \leq \ \$17.50 \end{aligned}$$

We have $D_q \ = \ [\$7.00, \$17.50]$.

Solution to $1c$***:***

$$\text{Revenue}r \ = R(p) \ = \ p \cdot q \ \ \rightsquigarrow \qquad\qquad R(p) \ = \ p(875 - 50p) \ = \ \boxed{875p - 50p^2)}$$

*You should note that the restriction $7 \leq p \leq 17.50$ is in force for **all** functions of the ticket price p, certainly this is so for $R(p)$.*

Solution to $1d$***:***

Note that nothing we did so far needed any knowledge of calculus.

To optimize R(p) we could either find p_0 for which the elasticity $E(p_0)$ is one or we examine the critical points of $R(p)$. We shall do the latter here:

$$R(p) \ = 875p - 50p^2 \ \ \rightsquigarrow \ \ R'(p) \ = \ 875 - 100p$$

which is zero exactly for $p_0 = 8.75$.

It's easy enough to see that 8.75 this is indeed a local max because $R''(p) = -100 > 0$ for any p, certainly so for $p = 8.75$. You'd still have to compute $R(8.75)$ and show that it exceeds $R(p)$ at the boundaries of the domain D_q, i.e., it is bigger than $R(7)$ or $R(17.50$. Why don't you verify such is indeed the case. Once you're back look at a different kind of reasoning:

The function $R(p) = -50p^2 + 875p$ is a second degree polynomial, i.e., a parabola. It is upside down because the factor -50 of p^2 is negative. It is clear that any such parabola has a global maximum at the one point where the tangent line is horizontal. That's all we need to conclude that revenue maxes out at a ticket price of \$8.75.

***Exercise 2**: A manufacturer wants to minimize the cost of the container it is building to hold its products. The container is a rectangular prism (a box) with a square base. It must have a volume of $10ft^3$ (cubic feet). The material that makes up the sides of the container costs \$0.12 per square foot. The material that makes up the top and bottom costs \$0.15 per square foot. What dimensions should the container have to minimize the cost of materials? How much will this optimal container cost?*

Hint: Use the volume equation to replace the variable for the height of the box with the variable for the length of the sides of the box.

Solution to 2:

The volume of the box is length $\times$ width $\times$ height. It has a square base, so length = width. Let's denote that by the letter "l" and let's denote the height by the letter "h". Then the volume v of the box is $v = l^2h = 10ft^3$. We follow the hint and use that equation to express h as a function of l:

$$\begin{aligned} l^2 \cdot h \ &= 10 \ \ \rightsquigarrow \ \ h \ = \ 10/l^2 \\ \text{cost } C(l) \ &= \text{cost of bottom } + \text{ cost of top } + \text{ cost of 4 sides } \ = \ .15 \cdot l^2 \ + \ .15 \cdot l^2 \ + \ 4 \cdot .12 \cdot l \cdot h \\ &= (3/10)l^2 \ + \ (48/100)l(10/l^2) \ = \ (3/10)l^2 \ + \ (48/10)l^{-1} \ = \ (3/10)(l^2 \ + \ 16l^{-1}) \end{aligned}$$

We compute $C'(l)$ and $C'(l)$ to find the critical points and check for min/max:

$$\begin{aligned} C'(l) \ &= (3/10)(2l \ - \ 16l^{-2}) \ = \ (3/5)(l \ - \ 8l^{-2})\,, \\ C\text{"}(l) \ &= (3/5)(1 \ - \ (-2)8l^{-3}), \ = \ (3/5)(1 \ + \ 16l^{-3}) \end{aligned}$$

Critical points:

$$\begin{aligned} C'(l) \ &= 0 \ \ \rightsquigarrow \ \ l \ = \ 8l^{-2} \ \ \rightsquigarrow \ \ l^3 \ = \ 8 \ \ \rightsquigarrow \ \ l \ = \ 2 \text{ feet}, \\ C\text{"}(2) \ &= (3/5)(1 \ + \ 16/2^3) \ = \ (3/5)(1 \ + \ 2) \ = \ (9/5) \ > \ 0 \end{aligned}$$

and it follows that there is a relative minimum at $l = 2$. How can we see that it is a global min? We look at the behavior of $C(l)$ for l at the interval boundaries 0 and ∞ of the domain $(0, \infty)$ of $C(l)$:

$$\begin{aligned} \lim_{l \to 0+} C(l) \ &= \ \lim_{l \to 0+} \Big((3/10)(l^2 \ + \ 16l^{-1})\Big) \ = \ \lim_{l \to 0+} \Big((3/10)(16l^{-1})\Big) \ = \ +\infty\,, \\ \lim_{l \to \infty} C(l) \ &= \ \lim_{l \to \infty} \Big((3/10)(l^2 \ + \ 16l^{-1})\Big) \ = \ +\infty \end{aligned}$$

We note that 2 is the only point with a horizontal tangent. Because of the limits above, there must be at least one minimum and any minimum, relative or absolute, must have a horizontal tangent. It follows that there is exactly one *minimum, hence it is absolute, and it is at the critical point $l = 2$. For the optimal container with base length 2 we get*

$$\begin{aligned} \text{cost } C(2) &= (3/10)(2^2 + 16/2) = 36/10 = \$3.60 \text{ per box} \\ \text{Height}: \ h &= 10/l^2 = 10/4 = 2.5 \text{ feet} \\ \text{Dimensions} &= (\text{length} = 2) \times (\text{width} = 2) \times (\text{height} = 2.5) \text{ cubic feet} \end{aligned}$$

Exercise 3: *A rectangular enclosure is to be built next to a river. There will be no fence on the river side of the enclosure. The cost for the material for the side parallel to the river is \$6.00 per foot. The material for the sides perpendicular to the river costs \$2.00 per foot. There is a budget of \$240.00 for the fence. What dimensions of the fence result in the largest enclosed area?*

Solution to 3:

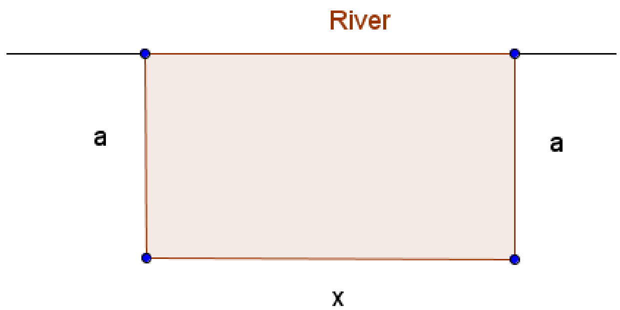

Figure 1: Problem 3: The enclosure at the river

Common sense tells us that we will not have fenced in the largest possible area if we did not spend all the money available: we could have used the unspent money to buy additional fencing for, say, the side parallel to the river, thus increasing x to $x + \Delta x$ and leaving a unchanged. This leads to an increase in area from $a \cdot x$ to the strictly larger amount $a(x + \Delta x)$. Now that we have established that the cost C equals \$240.00 we proceed as follows:

$$\text{Cost } C = 2a \cdot \$2.00 + x \cdot \$6.00 = \$240.00 \rightsquigarrow 4a + 6x = 240 \rightsquigarrow a = 60 - (3/2)x$$

This allows us to express the area as a function $A(x)$ of x only and examine its critical points to determine the maximum possible area:

$$\begin{aligned} \text{Area } A(x) &= a \cdot x = (60 - (3/2)x)x = -\frac{3}{2}x^2 + 60x \rightsquigarrow A'(x) = -3x + 60 \\ \text{Critical points: } A'(x) &= 0 \rightsquigarrow 3x = 60 \rightsquigarrow x = 20 \text{ feet} \end{aligned}$$

We have the same situation as in exercise 1. We look at the (one and only one) critical point of an upside down parabola and it follows that this only point with a horizontal tangent at the curve has a global maximum.

Alternative: use the 2nd derivative test: $A''(x) = -3$ is negative everywhere, specifically at the critical point $x = 20$, hence there is a local max. Now look what's happen at the boundary $x = 0$ and $x \to \infty$. First compute $A(0) = 0$ and verify it's smaller than $A(20) = 600$. Finally show that $\lim_{x\to\infty} A(x) = -\infty$ and you have shown that there is indeed a global max at $x = 20$.

We still must compute the dimensions of the optimal fence. We just computed $x = 20$. But we showed earlier that $a = 60 - (3/2)x$. So, $a = 60 - (3/2)20 = 30$.