Math 220 - Calculus f. Business and Management - Worksheet 31

Solutions for Worksheet 31 - Indefinite Integrals

Exercise 1:

Find the indefinite integrals of each of these functions

$$\textbf{1a}: f(x) = 3x \quad \textbf{1b}: f(x) = \sqrt{x} \quad \textbf{1c}: f(x) = x^2 + 2x + 5 \quad \textbf{1d}: f(x) = 3/\sqrt{x} \quad \textbf{1e}: f(x) = 4e^x$$

$$\textbf{1f}: f(x) = \frac{1}{3} + 3x - x^3 + \sqrt[3]{x} \quad \textbf{1g}: f(x) = \frac{1 - 3\sqrt[3]{u}}{u^2}$$

Solutions to #1:

$$\textbf{1a)}: \int 3x\,dx = 3 \cdot \frac{x^2}{2} + C = \boxed{\frac{3}{2}x^2 + C}$$

$$\textbf{1b)}: \int \sqrt{x}\,dx = \int x^{1/2}\,dx = \boxed{\frac{2}{3}x^{3/2} + C}$$

$$\textbf{1c)}: \int x^2 + 2x + 5\,dx = \frac{x^3}{3} + 2 \cdot \frac{x^2}{2} + 5x + C = \boxed{\frac{x^3}{3} + x^2 + 5x + C}$$

$$\textbf{1d)}: \int \frac{3}{\sqrt{x}}dx = 3\int x^{-1/2}\,dx = 3 \cdot 2 \cdot x^{1/2} + C = \boxed{6x^{1/2} + C}$$

$$\textbf{1e)}: \int 4e^x\,dx = \boxed{4e^x + C}$$

$$\textbf{1f)}: \int \left(\frac{1}{3} + 3x - x^3 + \sqrt[3]{x}\right)dx = \frac{1}{3}\int dx + 3\int x dx - \int x^3 dx + \int x^{1/3} dx$$

$$= \boxed{\frac{x}{3} + \frac{3}{2}x^2 - \frac{x^4}{4} + \frac{3}{4}x^{4/3} + C}$$

$$\textbf{1g)}: \int \frac{1 - 3\sqrt[3]{u}}{u^2}du = \int \frac{1}{u^2}du - 3\int \frac{u^{1/3}}{u^2}\,du = \int u^{-2}du - 3\int u^{-5/3}\,du$$

$$= (-1)u^{-1} - 3(-3/2)u^{-2/3} + C = \boxed{-1/u + (9/2)u^{-2/3} + C}$$

Exercise 2: *Find the specific solution $F(x) = \int(2x+3)\,dx$ that satisfies the condition $F(2) = 5$.*

***Solution to** 2:*

$$F(x) = \int (2x+3)\,dx = 2 \cdot x^2/2 + 3x + C = x^2 + 3x + C \;\; \textit{and}$$

$$F(2) = 5 \;\Rightarrow\; 4 + 6 + C = 5 \;\Rightarrow\; \boxed{C = -5}$$

Hence $F(x) = \boxed{x^2 + 3x - 5}$

Exercise 3: *Find $f(x)$ given that $f'(x) = 2 + 3/x$ and $f(1) = 4$.*

***Solution to** 3:*

$$f'(x) = 2 + 3/x \;\Rightarrow\; f(x) = \int 2 + 3/x\,dx \;\Rightarrow\; f(x) = 2x + 3\ln|x| + C$$

We now determine C. As $\ln(1) = 0$ we get

$$f(1) \;=\; 4 \;\;\Rightarrow\;\; 2(1) \,+\, 3(0) \,+\, C \;=\; 4 \;\;\Rightarrow\;\; C \;=\; 4 - 2 \;=\; 2.$$

Hence $f(x) = \boxed{2x \,+\, 3\ln|x| \,+\, 2}$

Exercise 4: *Solve the following three position/velocity/acceleration problems. Note that 4a, 4b and 4c are entirely separate problems!*

The first two problems deal with a falling body. Its acceleration caused by gravity is $-9.8m/sec^2$. (Note: acceleration is negative because it is pulling the body down. Positive velocity means something is going up).

4a: *If an object is thrown upward (from the ground) with a velocity of $15m/sec$, what will its velocity be after 2 seconds? Hint: acceleration is the derivative of velocity, so velocity is the integral of acceleration.*

4b: *Joan is on a platform 20 meters above the ground. How far above the ground will she be 2 seconds after she jumps? Hint: position is the integral of velocity.*

4c: *An object is moving with an initial position of $28m$ from the origin with an intial velocity of $-4m/sec$ and constant acceleration of $8m/sec^2$.*

How fast is the body moving after 2 seconds?
What is its position 3 seconds after the start?
When will it be $36m$ from the origin?

***Solution to** 4:*

The following is used for both part a and part b:

$$\text{Acceleration } a(t) \;=\; -\,9.8\; m/sec^2 \;\;\rightsquigarrow\;\; v(t) \;=\; \int a(t)dt \;=\; -9.8t \,+\, C$$

***Solution to** $4a$:*

Initial velocity was given as $v(0) \;=\; 15m/sec$. But we also have $v(0) \;=\; -9.8 \cdot 0 + C \;=\; C$. Both equations together give us $C = 15$ and this allows us to compute $v(2)$:

$$v(t) \;=\; -9.8t \,+\, 15 \;\;\rightsquigarrow\;\; v(2) \;=\; -19.6 \,+\, 15 \;=\; \boxed{-4.6\; m/sec}$$

***Solution to** $4b$:*

At time zero Joan is not in motion, hence $v(0) = 0$, and her position is on the platform, i.e., $s(0) = 20$.

$$v(t) \;=\; \int a(t)dt \;=\; -9.8t \,+\, C_1 \;\;\rightsquigarrow\;\; v(0) \;=\; -9.8 \cdot 0 \,+\, C_1 \;=\; C_1 \;\;\rightsquigarrow\;\; C_1 \;=\; 0 \;\;\rightsquigarrow\;\; v(t) \;=\; -9.8t$$

$$\text{Position } s(t) \;=\; \int v(t)dt \;=\; \int(-9.8t)dt \;=\; -4.9t^2 \,+\, C_2 \;\;\rightsquigarrow\;\; s(0) \;=\; -4.9 \cdot 0 \,+\, C_2 \;=\; C_2 \;\;\rightsquigarrow\;\; C_2 \;=\; 20$$

$$\rightsquigarrow\;\; s(t) \;=\; -\,4.9t^2 \,+\, 20 \;\;\rightsquigarrow\;\; s(2) \;=\; -4.9 \cdot 4 \,+\, 20 \;=\; 20 \,-\, 19.6 \;=\; .4$$

Joan is .4 meters above the ground.

Solution to $4c$***:***

$$v(t) \;=\; \int a(t)dt \;=\; \int 8dt \;=\; 8t\,+\,C_1 \;\;\rightsquigarrow\;\; v(0) \;=\; 8\cdot 0\,+\,C_1 \;=\; C_1$$

$$\rightsquigarrow\;\; C_1 \;=\; -\,4m/sec \;\;\rightsquigarrow\;\; v(t) \;=\; 8t-4 \;\;\rightsquigarrow\;\; v(2) \;=\; 16-4 \;=\; 12$$

$$s(t) \;=\; \int v(t)dt \;=\; \int (8t-4)dt \;=\; 4t^2\,-\,4t\,+\,C_2 \;\;\rightsquigarrow\;\; s(0) \;=\; 4\cdot 0\,-\,4\cdot 0 + C_2 \;=\; C_2 \;\;\rightsquigarrow\;\; C_2 \;=\; 28$$

$$\rightsquigarrow\;\; s(t) \;= 4t^2\,-\,4t\,+\,28 \;\;\rightsquigarrow\;\; s(3) \;=\; 4\cdot 9 - 4\cdot 3\,+\,28 \;=\; 36-12+28 \;=\; 52$$

To see when the object is 36 *m from the origin we solve the equation* $s(t) = 36$ *for* t*:*

$$s(t) \;= 4t^2\,-\,4t\,+\,28 \;=\; 36 \;\;\rightsquigarrow\;\; t^2\,-\,t\,+\,7 \;=\; 9 \;\;\rightsquigarrow\;\; t^2\,-\,t\,-\,2 \;=\; 0$$
$$\rightsquigarrow\;\; (t-2)\,(t+1) \;=\; 0 \;\;\rightsquigarrow\;\; t \;=\; -1, 2$$

Time cannot be negative so $s(t) = 36$ *only if* $t = 2$*.*

Exercise 5*: Solve these cost/revenue/profit questions.*

5a)*: The marginal revenue from selling item number* x *is* $6 + 2x + 1/x^2$*. The revenue from selling one item is* $\$40.00$*. Find the revenue function.*

5b)*: b) The marginal cost from selling the item* x *is* $4 + x + 2/x^3$*. The cost to produce one item is* $\$30.00$*. Find the cost function.*

5c)*: c) Use the information from a) and b) to find the profit function.*

Solution to $5a$***:***

$$\textit{Marginal revenue}\;\frac{dR}{dx} \;= 6\,+\,2x\,+\,1/x^2 \;\;\rightsquigarrow\;\; R(x) \;=\; \int (6\,+\,2x\,+\,x^{-2})dx \;=\; 6x\,+\,x^2\,-\,x^{-1}\,+\,C$$
$$R(1) \;=\; 40 \;\;\rightsquigarrow\;\; 6\cdot 1\,+\,1^2\,-\,1^{-1}\,+\,C \;=\; 40 \;\;\rightsquigarrow\;\; 6\,+\,C \;=\; 40 \;\;\rightsquigarrow\;\; C \;=\; 34$$
$$\rightsquigarrow\;\; R(x) \;=\; 6x\,+\,x^2\,-\,x^{-1}\,+\,34$$

Solution to $5b$***:***

$$\textit{Marginal cost}\;\frac{dC}{dx} \;= 4\,+\,x\,+\,2/x^3 \;\;\rightsquigarrow\;\; C(x) \;=\; \int (4\,+\,x\,+\,2x^{-3})dx$$
$$= 4x\,+\,x^2/2\,+\,2\cdot\frac{1}{-2}\cdot x^{-2}\,+\,C \;=\; 4x\,+\,x^2/2\,-\,x^{-2}\,+\,C$$
$$C(1) \;= 30 \;\;\rightsquigarrow\;\; 4\cdot 1\,+\,1^2/2\,-\,1^{-2}\,+\,C \;=\; 30 \;\;\rightsquigarrow\;\; 3.5\,+\,C \;=\; 30 \;\;\rightsquigarrow\;\; C \;=\; 26.5$$
$$\rightsquigarrow\;\; C(x) \;= 4x\,+\,x^2/2\,-\,x^{-2}\,+\,26.5$$

Solution to $5c$***:***

$$\begin{aligned}\textit{Profit}\;P(x) \;=\; R(x)\,-C(x) \;=\; & 6x + x^2\,-\,\frac{1}{x} \qquad\qquad +\,34\\ & -\,4x - \frac{x^2}{2} \qquad +\,\frac{1}{x^2}\,-\,26.5\\ =\; & 2x + \frac{x^2}{2}\,-\,\frac{1}{x}\,+\,\frac{1}{x^2}\,+\,7.5\end{aligned}$$