Math 220 - Calculus f. Business and Management - Worksheet 35

Solutions for Worksheet 35 - Definite Integrals

Solve these definite integrals by use of the power rule

***1**:* $\int_1^4 t^3\,dt$

Solution:

$$\int_1^4 t^3\,dt \;=\; \frac{t^4}{4}\bigg|_1^4 \;=\; \frac{4^4}{4} - \frac{1^4}{4} \;=\; \boxed{64 - \frac{1}{4}} \;=\; 63.75$$

***2**:* $\int_1^6 5^{-1/2}\,dx$

Solution:

$$\int_1^6 5^{-1/2}\,dx \;= 5^{-1/2}\int_1^6 x^0\,dx \;=\; 5^{-1/2}x\bigg|_1^6 \;=\; \frac{6}{\sqrt{5}} - \frac{1}{\sqrt{5}} \;=\; \frac{5}{\sqrt{5}} \;=\; \boxed{\sqrt{5}}$$

***3**:* $\int_1^3 \frac{1}{x}dx$

Solution:

$$\int_1^3 \frac{1}{x}dx \;=\; \ln|x|\bigg|_1^3 \;=\; \ln(3) - \ln(1) \;=\; \boxed{\ln(3)}$$

***4**:* $\int_{-2}^8 4e^{3t}\,dt$

Solution:

$$\int_{-2}^8 4e^{3t}\,dt \;=\; \frac{4}{3}\,e^{3t}\bigg|_{-2}^8 \;=\; \frac{4}{3}\,e^{24} \;-\; \frac{4}{3}\,e^{-6} \;=\; \boxed{\frac{4}{3}\Big(e^{24} \;-\; e^{-6}\Big)}$$

Solve these definite integrals using substitution

***5**:* $\int_2^5 x\cdot\sqrt[3]{x^2+4}\,dx$

***Solution:** Set $u = x^2+4$. Then, $\frac{du}{dx} = 2x$ so that $dx = \frac{du}{2x}$. Substitute into the original integral:*

$$\int_{x=2}^{x=5} x\cdot\sqrt[3]{u}\,\frac{du}{2x} \;=\frac{1}{2}\int_{u=8}^{u=29}\sqrt[3]{u}\,du \;=\; \frac{1}{2}\,\frac{3}{4}\,u^{4/3}\bigg|_8^{29} \;=\; \frac{3}{8}\cdot 29^{4/3} - \frac{3}{8}\cdot(\sqrt[3]{8})^4 \;=\; \boxed{\frac{3\cdot 29^{4/3}}{8} - 6}$$

The integration limits were tagged for clarity only with "$x = \dots$" and "$u = \dots$". This is not really "official" notation.

6: $\displaystyle\int_2^5 3te^{t^2+1}\,dt$

Solution: *Set* $u = t^2+1$. *Then* $\dfrac{du}{dt} = 2t$ *and* $dt = \dfrac{du}{2t}$. *As in the previous example we must be careful with the integration bounds: do they belong to the t-world or the u-world? You need not worry about this if you first compute the indefinite integral and only at the end match up the integration bounds with the integration variable.*
Substitute into the original:

$$\int 3te^{t^2+1}\,dt \;=\; \int 3t\;e^u\,\frac{du}{2t} \;=\; \frac{3}{2}\int\; e^u du \;=\; \frac{3}{2}\,e^u \;+\; C$$

We put back the integration bounds. That can either be done in the u-world

$$\int_2^5 3te^{t^2+1}\,dt \;=\frac{3}{2}\,e^u\bigg|_5^{26} \;=\; \boxed{\frac{3}{2}\,e^{26}-\frac{3}{2}\,e^5}$$

or in the t-world

$$\int_2^5 3te^{t^2+1}\,dt \;=\frac{3}{2}\,e^{t^2+1}\bigg|_2^{5} \;=\; \frac{3}{2}\,e^{5^2+1}-\frac{3}{2}\,e^{2^2+1} \;=\; \boxed{\frac{3}{2}\,e^{26}-\frac{3}{2}\,e^5}$$

The result is the same.

7: $\displaystyle\int_1^4 \frac{\ln(x)}{x}\,dx$

Solution: *Set* $u = \ln(x)$. *Then,* $\dfrac{du}{dx} = \dfrac{1}{x}$ *and therefore,* $dx = x\,du$. *We can now substitute back in and things will work out very nicely. Unless you "tag" the integration bounds as we did in problem #5, they should always match the integration variable. As soon as we switch from* dx *to* du *the bounds must switch as well:*

$$\int_1^4 \frac{\ln(x)}{x}\,dx \;=\; \int_0^{\ln(4)} \frac{u}{x}\cdot x\;du \;=\; \int_0^{\ln(4)} u\;du \;=\; \frac{u^2}{2}\bigg|_0^{\ln(4)} \;=\; \boxed{\frac{(\ln 4)^2}{2}}$$

Solve these definite integrals with partial integration

8: $\displaystyle\int_2^5 3te^{5t}\,dt$

Solution: *We set* $u = 3t$ *and* $dv = e^{5t}dt$. *Then,* $du = 3\cdot dt$ *and* $v = \dfrac{e^{5t}}{5}$. *We take the indefinite integral first and worry about evaluating it afterwards.*

$$uv - \int v\;du \;=(3t)(\frac{e^{5t}}{5}) \;-\; \int \frac{e^{5t}}{5}\cdot 3\cdot dt \;=\; \frac{3t}{5}e^{5t} \;-\; \frac{3}{5}\int e^{5t}dt \;=\; \frac{3t}{5}e^{5t} \;-\; \frac{3}{5}\cdot\frac{e^{5t}}{5} + C$$

You stand a high chance of making computational errors when doing partial integration. If time allows and accuracy is important you should consider differentiating this indefinite integral. If you did not make any mistakes you'll get back the original function $3te^{5t}$. *Observe that we do not bother with the constant* C. *We use the product rule:*

$$y \;=\frac{3t}{5}e^{5t} \;-\; \frac{3}{25}\cdot e^{5t} \;\;\Rightarrow\;\; y' \;=\; \left(\frac{3}{5}e^{5t} \;+\; \frac{3t}{5}\cdot 5e^{5t}\right) \;-\; \frac{3}{25}\cdot 5\cdot e^{5t} \;=\; \frac{3}{5}e^{5t} \;+\; 3te^{5t} \;-\; \frac{3}{5}e^{5t} \;=\; 3te^{5t}$$

Yes, we did get back the original function! We still must compute the definite integral. We need to bring back the integration bounds and evaluate:

$$\frac{3(5)}{5}e^{25} - \frac{3}{25}\cdot e^{25} - \left(\frac{3(2)}{5}e^{10} - \frac{3}{25}\cdot e^{10}\right) = \boxed{3e^{25} - \frac{3}{25}\cdot e^{25} - \frac{6}{5}\cdot e^{10} + \frac{3}{25}\cdot e^{10}}$$

9: $\displaystyle\int_1^4 \frac{\ln(x)}{x^3}\,dx$

Solution: *We choose* $u = \ln(x)$ *and* $dv = x^{-3}dx$. *Then* $du = \dfrac{dx}{x}$ *and* $v = \dfrac{-x^{-2}}{2}$. *We take the indefinite integral and then evaluate it after.*

$$\begin{aligned} uv - \int v\,du &= \frac{-\ln(x)x^{-2}}{2} - \int \frac{-x^{-2}}{2}\frac{dx}{x} = \frac{-\ln(x)}{2x^2} + \frac{1}{2}\int x^{-3}dx \\ &= \frac{-\ln(x)}{2x^2} + \frac{1}{2}\left(\frac{-1}{2}x^{-2}\right) + C = \frac{-\ln(x)}{2x^2} - \frac{x^{-2}}{4} + C = \frac{-\ln(x)}{2x^2} - \frac{1}{4x^2} + C \end{aligned}$$

As in the previous exercise we differentiate this indefinite integral and hope to get back the original function $\dfrac{\ln(x)}{x^3}$ *under the integral. We discard* C *and use again the product rule:*

$$\begin{aligned} y &= \frac{-\ln(x)}{2x^2} - \frac{1}{4x^2} = \frac{-\ln(x)}{2}x^{-2} - \frac{1}{4}x^{-2} \\ \Rightarrow\ y' &= \left(\frac{-1}{2x}x^{-2} + \frac{-\ln(x)}{2}\cdot(-2)x^{-3}\right) - \frac{1}{4}\cdot(-2)x^{-3} = \left(\frac{-1}{2x^3} + \frac{\ln(x)}{x^3}\right) + \frac{1}{2x^3} = \frac{\ln(x)}{x^3} \end{aligned}$$

Yes, we did get back the original function! We still must compute the definite integral. We need to bring back the integration bounds 4 *and* 1 *and evaluate:*

$$\int_1^4 \frac{\ln(x)}{x^3}dx = \frac{-\ln(4)}{2(4)^2} - \frac{1}{4(4)^2} - \left(\frac{-\ln(1)}{2(1)^2} - \frac{1}{4(1)^2)}\right) = \boxed{\frac{-\ln(4)}{32} - \frac{1}{64} + \frac{1}{4}}$$