Math 220 - Calculus f. Business and Management - Worksheet 36 - 37

Solutions for Worksheet 36 - 37 - Area and Average Value

Exercise 1:

Find the total distance travelled for the following situations:

$1a)$ *Velocity* $=$ 50 *kph constantly for* 4 *hours.*

$1b)$ *Velocity* $=$ 50 *kph for* 2 *hours,* 70 *kph for* 2 *hours.*

$1c)$ *Velocity* $=$ 50 *kph for* 1 *hour,* 60 *kph for* 2 *hours,* 40 *kph for* 1 *hour.*

Solution to $1a$***:*** *Distance* $s(4) = 50 \cdot 4$ *kph* $\cdot$ *hours* $= \boxed{200\,km}$

Solution to $1b$***:*** *Distance* $s(4) = 50 \cdot 2 + 70 \cdot 2$ *kph* $\cdot$ *hours* $= \boxed{240\,km}$

Solution to $1c$***:*** *Distance* $s(4) = 50 + 60 \cdot 2 + 40$ *kph* $\cdot$ *hours* $= \boxed{210\,km}$

Exercise 2: *Find the area between each of these curves and the x-axis:*

$2a)$ $\quad v(t) = t^2 \quad$ *from* $t = 0$ *to* $t = 4$

$2b)$ $\quad f(x) = x^2 + 3x + 7 \quad$ *from* $x = 1$ *to* $x = 5$

Solution to $2a$***:*** *We note that* $t^2 \geq 0$ *for all* t *and we need not worry about breaking up the interval of integration into several segments.*

$$\text{Area}[0,4] = \int_0^4 t^2 dt = 1/3\, t^3\Big|_0^4 = 4^3/3 - 0 = \boxed{64/3}$$

Solution to $2b$***:*** *We note that both* x^2 *and* $3x$ *are positive for* $1 \leq x \leq 5$*. It follows that the same is true for* $f(x)$ *and we need not worry about breaking up the interval of integration into several segments.*

$$\begin{aligned}\text{Area}[1,5] &= \int_1^5 \big(x^2 + 3x + 7\big)dx = \big(1/3\, x^3 + 3/2x^2 + 7x\big)\Big|_1^5 \\ &= \frac{125}{3} + \frac{75}{2} + 35 - \frac{1}{3} - \frac{3}{2} - 7 = \frac{124}{3} + \frac{72}{2} + 28 = \boxed{\frac{124}{3} + 64}\end{aligned}$$

Exercise 3: *Find both the signed area and the area between each of these curves and the* x*-axis:*

$3a)$ $\quad f(x) = x^3 \quad$ *from* $x = -1$ *to* $x = 1$

$3b)$ $\quad h(x) = x^2 - 6x + 5 \quad$ *from* $x = 1$ *to* $x = 5$ *and from* $x = 0$ *to* $x = 6$

Solution to $3a$***:*** *Antiderivative* $\int x^3 dx = x^4/4 + C$*. It is clear that* $x^3 < 0$ *for* $x < 0$ *and* $x^3 > 0$ *for* $x > 0$*. We* *must split the interval of integration at* $x = 0$.

$$\begin{aligned} x^4/4\Big|_{-1}^{0} &= 0 - 1/4 \ = -1/4 \\ x^4/4\Big|_{0}^{1} &= 1/4 - 0 \ = 1/4 \\ \textit{Area}[-1,1] &= 1/4 + 1/4 = 1/2 \\ \textit{Signed area}[-1,1] &= -1/4 + 1/4 = 0 \end{aligned}$$

Solution to $3b$***:*** $h(x) = x^2 - 6x + 5 = (x-5)(x-1)$ *and* $h(x) > 0$ *for* $x < 1$, $h(x) < 0$ *for* $1 < x < 5$, $h(x) > 0$ *for* $x > 5$. *Antiderivative* $\int x^2 - 6x + 5dx = x^3/3 - 3x^2 + 5x + C$.

$$\begin{aligned} x^3/3 - 3x^2 + 5x\Big|_{0}^{1} &= 7/3 \\ x^3/3 - 3x^2 + 5x\Big|_{1}^{5} &= -32/3 \\ x^3/3 - 3x^2 + 5x\Big|_{5}^{6} &= 7/3 \end{aligned}$$

We now can compute all signed and unsigned areas:

$$\begin{aligned} \textit{Area}[1,5] &= |-32/3| \ = \ 32/3 \\ \textit{Area}[0,6] &= |7/3| \ + \ |-32/3| \ + \ |7/3| \ = \ 46/3 \\ \textit{Signed area}[1,5] &= -32/3 \\ \textit{Signed area}[0,6] &= 7/3 \ - \ 32/3 \ + \ 7/3 \ = \ -18/3 = -6 \end{aligned}$$

Exercise 4*:*

Find the area between these non-intersecting curves (how can you show they don't intersect?):

$4a)$ $\quad f(x) = x^2 + 10$ *and* $g(x) = 2x$ *from* $x = 0$ *to* $x = 5$

$4b)$ $\quad f(x) = 2x - 3$ *and* $g(x) = 5x + 7$ *from* $x = 1$ *to* $x = 4$

Solution to $4a$***:*** *We note that* $f(x) \geq 10$ *everywhere and* $f(5) = 35 > 10$. *On the other hand we have* $g(x) < 10$ *for* $0 \leq x < 5$ *and* $g(5) = 10$. *It follows that the graph of* $f(\cdot)$ *is strictly above the one of* $g(\cdot)$ *on the interval* $[0,5]$ *of integration and it follows that the curves do not intersect.*

$$\textit{Area}[0,5] \ = \ \int_0^5 \big(x^2 + 10 - 2x\big)dx \ = \ \big(\frac{x^3}{3} - x^2 + 10x\big)\Big|_0^5 \ = \ 200/3$$

Solution to $4b$***:*** *On the interval of integration* $[1,4]$ *we have* $f(x) \leq f(4) = 11$ *and* $g(x) \geq g(1) = 12$. *It follows that the graph of* $g(\cdot)$ *is strictly above the one of* $f(\cdot)$ *and it follows that the curves do not intersect.*

$$\textit{Area}[1,4] \ = \ \int_1^4 \big(5x + 7 - 2x + 3\big)dx \ = \ \int_1^4 \big(3x + 10\big)dx \ = \ \big(\frac{3x^2}{2} + 10x\big)\Big|_1^4 \ = \ 552.5$$

Exercise 5*:*

Find the area between these curves:

$5a)$ $\quad f(x) = 5x - 2$ *and* $g(x) = 4x + 1$ *from* $x = 2$ *to* $x = 6$

$5b)$ $\quad f(x) = x^2 - 4x + 7$ *and* $g(x) = -x^2 + 4x + 1$ *(Note: you will need to find the integration bounds.)*

***Solution to** 5a:* *The area we want is the same as the area between $f(x) - g(x) = x - 3$ and the x-axis and $x = 2$ and $x = 6$. $f(x) - g(x)$ is negative for $x < 3$ and positive for $x > 3$. We must split the integral at $x = 3$:*

$$\begin{aligned} \textit{Area}[2,6] &= -\int_2^3 \big(x-3\big)dx + \int_3^6 \big(x-3\big)dx \\ &= -\Big(\frac{x^2}{2} - 3x\Big)\Big|_2^3 + \Big(\frac{x^2}{2} - 3x\Big)\Big|_3^6 = 2 + 4.5 = 6.5 \end{aligned}$$

***Solution to** 5b:* *We set $h(x) = f(x) - g(x) = 2x^2 - 8x + 6$. Our task is to find the area between $h(x)$ and the x-axis, between the x-values where $f(x) = g(x)$, i.e., $h(x) = 0$:*

$$h(x) = 2\big(x^2 - 4x + 3\big) = 2(x-3)(x-1) = 0 \quad \rightsquigarrow \ x = 3,\ x = 1.$$

We must integrate between $1 \le x \le 3$. On this interval $h(x)$ is negative (check this by plugging in $x = 2$). Hence

$$\textit{Area}[1,3] = -\int_1^3 \big(2x^2 - 8x + 6\big)dx = -\Big(\frac{2x^3}{3} - 4x^2 + 6x\Big)\Big|_1^3 = \frac{8}{3}$$

***Exercise 6**:*

Find the average velocity for the situations in problem 1.

***Solution to** 6:* *In each instance the average velocity is $s(4)/4$, Hence*

$$\textbf{\textit{1a}} \rightsquigarrow \frac{200}{4} = 50\ \textit{kph}, \quad \textbf{\textit{1b}} \rightsquigarrow \frac{240}{4} = 60\ \textit{kph}, \quad \textbf{\textit{1c}} \rightsquigarrow \frac{210}{4} = 52.5\ \textit{kph}.$$

***Exercise 7**:*

The price for a product is increasing over time according to this formula: $p(t) = 2e^{.01t}$ where t is measured in weeks. What is the average price from week 2 to week 5?

***Solution to** 7:*

$$\textit{Average price}[2,5] = \frac{1}{3}\int_2^5 p(t)dt = \frac{2}{3}\int_2^5 e^{.01t}dt = \frac{2}{3\cdot .01}e^{.01t}\Big|_2^5 = \frac{200}{3}\big(e^{.05} - e^{.02}\big).$$