Math 220 - Calculus f. Business and Management - Worksheet 39

Solutions for Worksheet 39 - Improper Integrals and Capital Value

Exercise 1:

Determine whether each of these integrals converges or diverges. If the integral converges, find its value.

$1a)\qquad \displaystyle\int_1^\infty \frac{3}{\sqrt{x}}\,dx$

$1b)\qquad \displaystyle\int_1^\infty \frac{2x}{(x^2+3)^3}\,dx$

$1c)\qquad \displaystyle\int_{-\infty}^0 3x^2 3^{x^3}\,dx$

$1d)\qquad \displaystyle\int_{-\infty}^\infty (x+2)dx$

Solution to $1a$***:***

$$\int_1^\infty \frac{3}{\sqrt{x}}\,dx \;=3\lim_{T\to\infty}\int_1^T x^{-1/2}\,dx \;=\; 3\lim_{T\to\infty}\big(2x^{1/2}\big)\Big|_1^T \;=\; 6\lim_{T\to\infty}T^{1/2}-6 \;=\; \infty$$

because $T^{1/2}\to\infty$ *as* $T\to\infty$. *The integral diverges.*

Solution to $1b$***:*** *Substitute* $u=x^2+3 \;\rightsquigarrow\; du=2xdx$. *Further* $x=1 \;\rightsquigarrow\; u=4$, $x=\infty \;\rightsquigarrow\; u=\infty$*:*

$$\begin{aligned}\int_1^\infty \frac{2x}{(x^2+3)^3}\,dx \;&=\; \lim_{T\to\infty}\int_4^T u^{-3}\,du \;=\; \lim_{T\to\infty}\big(\frac{-1}{2}u^{-2}\big)\Big|_4^T\\ &=\frac{1}{2}\big(4^{-2}-\lim_{T\to\infty}T^{-2}\big) \;=\; \frac{1}{32}-\frac{1}{2}\cdot 0 \;=\; \frac{1}{32}\end{aligned}$$

because $T^{-2}\to 0$ *as* $T\to\infty$.

Solution to $1c$***:*** *Substitute* $u=x^3 \;\rightsquigarrow\; du=3x^2dx$. *Further* $x=0 \;\rightsquigarrow\; u=0$, $x=-\infty \;\rightsquigarrow\; u=-\infty$*:*

$$\int_{-\infty}^0 3x^2 3^{x^3}dx \;=\; \int_{-\infty}^0 3^u du \;=\; \lim_{T\to-\infty}\int_T^0 3^u du$$

Note that $\int 3^u du \;=\; (1/ln3)3^u + C$ *and* $3^u\to 0$ *as* $T\to-\infty$. *We get*

$$\lim_{T\to-\infty}\int_T^0 3^u du \;=\; \frac{1}{ln3}\lim_{T\to-\infty}3^u\Big|_T^0 \;=\; \frac{1}{ln3}\big(1-\lim_{T\to-\infty}3^T\big) \;=\; \frac{1}{ln3}.$$

Solution to $1d$***:***

$$\begin{aligned}\int_{-\infty}^\infty (x+2)dx \;&=\; \lim_{R\to-\infty}\int_R^0 (x+2)\,dx \;+\; \lim_{T\to\infty}\int_0^T (x+2)\,dx\\ &=\lim_{R\to-\infty}\big(\frac{x^2}{2}+2x\big)\Big|_R^0 \;+\; \lim_{T\to\infty}\big(\frac{x^2}{2}+2x\big)\Big|_0^T\\ &=0-\lim_{R\to-\infty}\big(\frac{R^2}{2}+2R\big) \;+\; \lim_{T\to\infty}\big(\frac{T^2}{2}+2T\big).\end{aligned}$$

The integral diverges and there is no limit because the last expression is of the form "$-\infty + \infty$".

Note for the following exercises that capital value is explained on p.285 of the text.

***Exercise 2**: A man about to retire wants to purchase an annuity that will pay \$5000 per year for the rest of his life. In order to be sure that he never loses his income no matter how old he lives to be, he wants to invest enough money that the annuity will pay \$5000 per year forever. At an interest rate of 6%, how much will the annuity cost him?.*

Solution to 2:

The investment horizon is ∞. We compute the capital value of the investment:

$$\begin{aligned} \textit{Capital value} \; &= \int_0^\infty \big(5000e^{-.06t}\big)dt \; = \; \frac{5000}{-.06} \lim_{T\to\infty} \; e^{-.06t}\Big|_0^T \\ &= -\,83333.33\,\big(\lim_{T\to\infty} \; e^{-.06T} - 1\big) \; = \; -83333.33\,(0-1) \; = \; \$83333.33 \end{aligned}$$

***Exercise 3**: Another retiree also wants an annuity. In this case she wants an annuity that will grow over time to keep up with inflation. How much will an annuity cost that will pay $\$5000e^{.01t}$ forever?*

Solution to 3:

ASSUMPTION: the interest rate on which the discounting of the annuity is based is again 6%. Without such a rate which cheapens the buying price nobody would buy the annuity because the money for the purchase could be placed into an interest bearing account instead.

The investment horizon is ∞. We compute the capital value of the investment:

$$\begin{aligned} \textit{Capital value} \; &= \int_0^\infty \big(5000e^{.01t}e^{-.06t}\big)dt \; = \; \frac{5000}{-.05} \lim_{T\to\infty} \; e^{-.05t}\Big|_0^T \\ &= -\,100,000\,\big(\lim_{T\to\infty} \; e^{-.05T} - 1\big) \; = \; -100,000\,(0-1) \; = \; \$100,000.00 \end{aligned}$$