Math 220 - Calculus f. Business and Mgmt - Worksheet 43

Solutions for Worksheet 43 - Local Extrema with Two Variables

Exercise 1:

Find any minima, maxima or inflection point for this function: $f(x) = x^3 - 5x^2 - 8x + 4$.

***Solution to** 1**:** Note that this is a function of only **ONE** variable and you solve this using the techniques you learnt earlier in the course.*

$$\begin{aligned} f'(x) &= 3x^2 - 10x - 8 \;=\; 0 \;\rightsquigarrow\; x \;=\; \frac{1}{2 \cdot 3}\Big(-(-10) \pm \sqrt{(-10)^2 - 4 \cdot 3 \cdot (-8)}\Big) \\ &= \frac{1}{6}\big(10 \pm \sqrt{100 + 96}\big) \;=\; \frac{1}{6}\big(10 \pm 14\big) \;\rightsquigarrow\; x = 4 \text{ or } x = -\frac{2}{3}. \end{aligned}$$

It follows that the critical points are $x = 4, x = -2/3$. *There are no boundary points to worry about because* $D_f = (-\infty, \infty)$.

We now examine $f''(x)$ *at those two critical points.*

$$\begin{aligned} f''(x) \;=\; 6x - 10 \;\rightsquigarrow\; f''(4) &= 24 - 10 > 0 \;\rightsquigarrow\; \textit{local minimum at } x = 4, \\ f''(-2/3) &= -4 - 10 < 0 \;\rightsquigarrow\; \textit{local maximum at } x = -2/3. \end{aligned}$$

Exercise 2:

Find any local extrema or saddle points for the following functions (Note: These functions are taken from the exercises at the end of Section 28 in your textbook):

$2a)$ $f(x,y) \;=\; x^2 - 2xy + 2y^2 + x$,

$2b)$ $f(x,y) \;=\; (x-1)^2 + y^3 - y^2 - y + 1$,

$2c)$ $f(x,y) \;=\; x^2 + y^2 - y^2x^2 - 4$,

$2d)$ $f(x,y) \;=\; x^2 + 4y^3 - 6xy - 1$.

***Solution to** $2a$**:** We compute* f_x *and* f_y *to locate the critical points:*

$$\begin{aligned} \textbf{\#1:} &\quad f_x(x,y) \;=\; 2x - 2y + 1 \;=\; 0 \\ \textbf{\#2:} &\quad f_y(x,y) \;=\; -2x + 4y \;=\; 0 \\ \textit{add } \textbf{\#1+\#2:} &\quad 2y + 1 \;=\; 0 \;\rightsquigarrow\; \boxed{y = \frac{-1}{2}} \\ \textit{plug } y \textit{ into } \textbf{\#1:} &\quad 2x + 1 + 1 \;=\; 0 \;\rightsquigarrow\; \boxed{x = -1} \end{aligned}$$

There is only one critical point with coordinates $(-1, -1/2)$. *We examine its nature with help of the second order partials:*

$$\begin{aligned} & f_{xx}(x,y) \;= 2, \quad f_{xy} \;=\; -2, \quad f_{yx}(x,y) \;=\; -2, \quad f_{yy} \;=\; 4, \\ & \qquad \rightsquigarrow D\big(-1, -\frac{1}{2}\big) \;=\; 2 \cdot 4 - (-2)^2 \;=\; 4 > 0 \;\rightsquigarrow\; \textit{local extremum at } \big(-1, -\frac{1}{2}\big). \\ & f_{xx}\big(-1, -\frac{1}{2}\big) \;= 2 \;> 0 \;\rightsquigarrow\; \textit{local minimum at } \big(-1, -\frac{1}{2}\big). \end{aligned}$$

***Solution to** 2b: We compute f_x and f_y to locate the critical points:*

$$\begin{aligned} f_x(x,y) &= 2(x-1) \;=\; 0 \;\rightsquigarrow\; \boxed{x=1} \\ f_y(x,y) &= 3y^2-2y-1 \;=\; 0 \;\rightsquigarrow\; x \;=\; \frac{1}{2\cdot 3}\Big(-(-2) \pm \sqrt{4-4\cdot 3(-1)}\Big) \\ &= \frac{1}{6}\Big(2 \pm \sqrt{16}\Big) \;\rightsquigarrow\; \boxed{y=1 \text{ or } y=-\frac{1}{3}} \end{aligned}$$

There are two critical points with coordinates $(1,1)$ and $(1,-1/3)$. We examine them with help of the second order partials:

$$\begin{aligned} f_{xx}(x,y) &= 2, \quad f_{xy} \;=\; 0, \quad f_{yx}(x,y) \;=\; 0, \quad f_{yy} \;=\; 6y-2, \\ &\rightsquigarrow D(x,y) \;=\; 2(6y-2) \;=\; 4(3y-1) \end{aligned}$$

$$\begin{aligned} x=1, y=1: &\quad D \;=\; 8 \;>\; 0 \;\rightsquigarrow\; \text{local extremum};\; f_{xx}(1,1) \;=\; 2 \;>\; 0 \;\rightsquigarrow\; \text{local minimum}. \\ x=1, y=-\frac{1}{3}: &\quad D \;=\; -8 \;<\; 0 \;\rightsquigarrow\; \text{saddle point}. \end{aligned}$$

***Solution to** 2c: We compute f_x and f_y to locate the critical points:*

$$\begin{cases} f_x(x,y) \;=\; 2x-2xy^2) \;=\; 2x(1-y^2) \;=\; 0 \\ f_y(x,y) \;=\; 2y-2x^2y \;=\; 2y(1-x^2) \;=\; 0 \end{cases} \rightsquigarrow\; \boxed{x=0 \text{ or } x \pm 1}, \quad \boxed{y=0 \text{ or } y \pm 1}$$

and we have a total of nine critical points. We examine them with help of the second order partials:

$$\begin{aligned} f_{xx}(x,y) &= 2-2y^2, \quad f_{xy} \;=\; -4xy, \quad f_{yx}(x,y) \;=\; -4xy, \quad f_{yy} \;=\; 2-2x^2, \\ &\rightsquigarrow D(x,y) \;=\; 2(1-y^2)2(1-x^2)-16x^2y^2 \;=\; 4\big((1-x^2)(1-y^2)-4x^2y^2\big). \\ x=0, y=0 &\rightsquigarrow D(x,y) \;=\; 4>0 \;\rightsquigarrow\; \text{local extremum};\, f_{xx}(0,0)=2>0 \;\rightsquigarrow\; \text{local minimum}. \\ x=\pm 1, y=\pm 1 &\rightsquigarrow D(x,y) \;=\; 4(0-4)<0 \;\rightsquigarrow\; \text{saddle point}; \\ x=0, y=\pm 1 &\rightsquigarrow D(x,y) \;=\; 4(0-0)=0 \;\rightsquigarrow\; \text{unknown}; \\ x=\pm 1, y=0 &\rightsquigarrow D(x,y) \;=\; 4(0-0)=0 \;\rightsquigarrow\; \text{unknown}. \end{aligned}$$

***Solution to** 2d: We compute f_x and f_y to locate the critical points:*

$$\begin{cases} f_x \;=\; 2x-6y \;=\; 0 \\ f_y \;=\; 12y^2-6x \;=\; 0 \end{cases} \rightsquigarrow \quad \begin{cases} x=3y, \\ 12y^2-18y \text{ (because } x=3y) \;=\; 0 \end{cases}$$

$$\rightsquigarrow\; 6y(2y-3) \;=\; 0 \;\rightsquigarrow\; y=0 \text{ or } y=3/2.$$

We use $x=3y$ to find x for those two y: If $y=0$ then $x=0$. If $y=3/2$ then $x=9/2$.
The critical points are $\boxed{\text{(0,0), (9/2, 3/2)}}$. We examine them with help of the second order partials:

$$\begin{aligned} f_{xx} &= 2, \quad f_{xy} \;=\; -6, \quad f_{yx} \;=\; -6, \quad f_{yy} \;=\; 24y \\ &\rightsquigarrow D(x,y) \;=\; 48y-36 \;=\; 12(4y-3). \\ x \;=0, y \;=\; 0 &\rightsquigarrow D \;=\; -36<0 \;\rightsquigarrow\; \boxed{\text{saddle point at } (0,0)}; \\ x \;=\; \frac{9}{2}, y \;=\; \frac{3}{2} &\rightsquigarrow D \;=\; 12(6-3)>0 \;\rightsquigarrow\; \text{local extremum}; \\ f_{xx}\Big(\frac{9}{2},\frac{3}{2}\Big) &= 2>0 \;\rightsquigarrow\; \boxed{\text{local minimum at } \Big(\frac{9}{2},\frac{3}{2}\Big)}. \end{aligned}$$