Math 220 - Calculus f. Business and Management - Worksheet 44-45

Solutions for Worksheet 44 - 45 - LaGrange Multipliers

Exercise 1:

A rectangular enclosure is to be built. The East and West sides cost $\$6$ *per meter. The North and South sides cost* $\$5$ *per meter. What is the biggest area that can be achieved for* $\$1,500$*?*

Find the lowest cost option for this situation applying the method learned in earlier optimization problems (find critical points of a function of a single variable ...)

Solution to 1***:*** *The cost for the fence is* $\$(2x)5.00$ *for the north and south sides plus* $\$(2y)6.00$ *for the east and west sides. We must maximize the area* $A(x,y) = xy$ *subject to the cost constraint*
$C(x,y) = 10x + 12y = 1,500$.

$$\begin{aligned}
10x + 12 &= 1,500 \rightsquigarrow y = 125 - \frac{5}{6}x; \\
f(x) &= A(x,y) = x\big(125 - \frac{5}{6}\big)x = 125x - \frac{5}{6}x^2 \\
&\rightsquigarrow f'(x) = 125 - \frac{5}{3}x
\end{aligned}$$

We note that f(x) is an upside-down parabola ($-5/3 < 0$*) with a vertical axis of symmetry. It follows that if the critical point* $x_{\max}$ *for which* $f'(x_{\max}) = 0$ *is inside the domain* $D_f = [0, \infty)$ *then the area function has an absolute maximum there.*

$$\begin{aligned}
f'(x) &= 125 - \frac{5}{3}x = 0 \rightsquigarrow 25 = \frac{x}{3} \rightsquigarrow \boxed{x = 75}; \\
&\rightsquigarrow y = 125 - \frac{5 \cdot 75}{6} = 125 - \frac{5 \cdot 25 \cdot 3}{2 \cdot 3} = 125 - \frac{125}{2} = \frac{125}{2} \rightsquigarrow \boxed{y = \frac{125}{2}}
\end{aligned}$$

We have already established that $f(x)$ *indeed does have a global maximum at* $x = 75$ *meters. Hence the maximum area that can be fenced is* $A(75, 125/2) = 4687.5$ *square meters.*

Exercise 2:

Solve the problem in exercise 1 using LaGrange multipliers instead.

Solution to 2***:*** *The task is to maximize the area* $A(x,y) = xy$ *subject to the constraint*

$g(x,y) = C(x,y) - 1,500 = 10x + 12y - 1,500 \ = \ 0$ *(see exercise 1).*

$$\textit{Lagrange function:} \quad F(x,y,\lambda) \ = \ A(x,y) + \lambda g(x,y) \ = \ xy + 10\lambda x + 12\lambda y - 1,500\lambda$$

$$\rightsquigarrow \begin{cases} \textbf{a)} & F_x \ = \ y + 10\lambda \ = \ 0, \\ \textbf{b)} & F_y \ = \ x + 12\lambda \ = \ 0, \\ \textbf{c)} & F_\lambda \ = \ 10x + 12y - 1,500 \ = \ 0 \end{cases}$$

multiply a) with $12/10 \ \rightsquigarrow$ ***d)*** $\dfrac{12y}{10} + 12\lambda \ = \ 0$

subtract d) - b) $\rightsquigarrow$ ***e)*** $\dfrac{12y}{10} - x \ = \ 0 \ \rightsquigarrow \ x \ = \ \dfrac{12y}{10}$

plug that into c): $10 \cdot \dfrac{12y}{10} + 12y - 1,500 \ = \ 0 \ \rightsquigarrow \ 24y \ = \ 1,500 \ \rightsquigarrow \ \boxed{y \ = \ \dfrac{125}{2}}$

solve e) for x*:* $x \ = \ \dfrac{12y}{10} \ = \ \dfrac{12 \cdot 125}{10 \cdot 2} \ = \ \dfrac{3 \cdot 4 \cdot 25 \cdot 5}{2 \cdot 5 \cdot 2} \ = \ 25 \cdot 3 \ \rightsquigarrow \ \boxed{x \ = \ 75}$

We have obtained the same result as in exercise 1.

Exercise 3*:*

Solve these problems from your textbook using LaGrange Multipliers: Problems 4, 6 and 8

Exercise 3-A - *Textbook problem 4:*

A company has two plants that produce diamond necklaces. At plant A, it costs $x^2 + 1,200$ *dollars to make* x *necklaces. At plant B, it costs* $3y^2 + 800$ *dollars to make* y *necklaces. An order has come to the company for* $1,200$ *necklaces. (a) How many necklaces should be made in plant A and how many necklaces should be made in plant B if the company wishes to minimize the cost? (b) If the company charges the customer* $\$1,000$ *for each necklace, how much profit will they make from this order?*

Solution to **3*-A:*** *We must minimize cost* $C(x,y) = x^2 + 1,200 + 3y^2 + 800 = x^2 + 3y^2 + 2,000$ *given the fixed quantity* $x + y = 1,200$*. Hence the constraint function is* $g(x,y) = x + y - 1,200 = 0$*.*

$$\textit{Lagrange function:} \quad F(x,y,\lambda) \ = \ C(x,y) + \lambda g(x,y) \ = \ x^2 + 3y^2 + 2,000 + \lambda x + \lambda y - 1,200\lambda$$

$$\rightsquigarrow \begin{cases} \textbf{a)} & F_x \ = \ 2x + \lambda \ = \ 0, \\ \textbf{b)} & F_y \ = \ 6y + \lambda \ = \ 0, \\ \textbf{c)} & F_\lambda \ = \ x + y - 1,200 \ = \ 0 \end{cases}$$

subtract a) - b) $\rightsquigarrow \ 2x - 6y \ = \ 0 \ \rightsquigarrow \ \boxed{x = 3y}$

plug that into c) $\rightsquigarrow \ F_\lambda \ = \ 3y + y - 1,200 \ = \ 0 \ \rightsquigarrow \ \boxed{y = 300}$

use again $x = 3y \ \rightsquigarrow \ \boxed{x = 900}$

It follows that the optimal allocation of resources is 900 *necklaces at plant A and* 300 *necklaces at plant B. To compute the profit we observe that revenue is* $\$1,000$ *per necklace* $\times 1,200$ *necklaces, i.e., revenue is* $\$1,200,000$*. Hence the profit is revenue minus* $C(900,300)$*:*

$$\begin{aligned} 1,200,000 - (900^2 + 3 \cdot 300^2 + 2,000) \ &= \ 1,200,000 - 900 \cdot 900 - (3 \cdot 300)300 - 2,000 \\ &= 1,000 \cdot 1,200 - 900(900 + 300) - 2,000 \ = \ (1,000 - 900)1,200 - 2,000 \ = \ \boxed{\$118,000.00} \end{aligned}$$

Exercise 3-B - *Textbook problem 6:*

The total cost to produce x widgets and y bidgets is given by $C(x,y) = 3x^2 + 4y^2 + 2xy + 3$. If a total of ten items must be made, how should production be allocated so that total cost is minimized?

Solution to* 3*-B: *The task is to minimize the cost $C(x,y) = 3x^2 + 4y^2 + 2xy + 3$ subject to the constraint $g(x,y) = x + y - 10 = 0$.*

$$\text{Lagrange function:}\quad F(x,y,\lambda) = C(x,y) + \lambda g(x,y) = 3x^2 + 4y^2 + 2xy + 3 + \lambda x + \lambda y - 10\lambda$$

$$\rightsquigarrow \begin{cases} \textbf{a)} & F_x = 6x + 2y + \lambda = 0, \\ \textbf{b)} & F_y = 8y + 2x + \lambda = 0, \\ \textbf{c)} & F_\lambda = x + y - 10 = 0 \end{cases}$$

$$\text{subtract a) - b)} \rightsquigarrow 4x - 6y = 0 \rightsquigarrow x = \frac{3y}{2}$$

$$\text{plug that into c):}\quad \frac{3y}{2} + y - 10 = 0 \rightsquigarrow \frac{5y}{2} = 10 \rightsquigarrow \boxed{y = 4}$$

$$\text{use } x + y = 10 \rightsquigarrow \boxed{x = 6}$$

The factory must produce 6 widgets and 4 bidgets to minimize production cost.

Exercise 3-C *- Textbook problem 8:*

Find the dimensions that will minimize the surface area (and hence the cost) of a rectangular tank, open on top, with a volume of 32 cubic feet.

Hint: There are 4 variables (including λ) in this problem.
a) Solve F_x and F_y for λ and set the results equal to each other and solve for y.
b) Use the solution for x in F_z to find λ.
c) use F_y and F_λ to finish the problem

Solution to* 3*-C: *We denote the width by x, the length by y and the height by z. The surface area consists of two rectangular sides of area xz, two rectangular sides of area yz, and the bottom rectangle of area xy. We want to minimize the total surface area $S(x,y,z) = xy + 2xz + 2yz$ square feet given that volume $V(x,y,z) = xyz = 32$ cubic feet, i.e., $xyz - 32 = 0$. Hence the constraint function is $g(x,y,z) = xyz - 32$. We may assume $x,y,z > 0$ because no dimension can be negative and if anyone of them is zero then the constraint $xyz - 32 = 0$ is not satisfied. It follows that there are no problems multiplying or dividing by any of x, y, z is unproblematic.*

$$\text{Lagrange function:}\quad F(x,y,z,\lambda) = S(x,y,z) + \lambda g(x,y,z) = xy + 2xz + 2yz + \lambda xyz - 32\lambda$$

$$\rightsquigarrow \begin{cases} \textbf{a)} & F_x = y + 2z + \lambda yz = 0, \\ \textbf{b)} & F_y = x + 2z + \lambda xz = 0, \\ \textbf{c)} & F_z = 2x + 2y + \lambda xy = 0, \\ \textbf{d)} & F_\lambda = xyz - 32 = 0 \end{cases}$$

Step 1) Solve F_x and F_y for λ and set the results equal to each other and solve for y:

$$\textbf{a)} \rightsquigarrow \lambda yz = -(y + 2z) \rightsquigarrow \lambda = -\frac{y + 2z}{yz},$$

$$\textbf{b)} \rightsquigarrow \lambda xz = -(x + 2z) \rightsquigarrow \lambda = -\frac{x + 2z}{xz},$$

$$\text{equate those two:}\quad \frac{y + 2z}{yz} = \frac{x + 2z}{xz} \rightsquigarrow xyz + 2xz^2 = xyz + 2yz^2 \rightsquigarrow \boxed{x = y}$$

Step 2) Use the solution for x in F_z to find λ.

$$F_z \;= 2x + 2x + \lambda x^2 \;=\; 0 \;\rightsquigarrow\; x(4 + \lambda x) \;=\; 0 \;\rightsquigarrow\; \boxed{\lambda = -\frac{4}{x}}$$

Step 3) plug λ into F_y and use F_λ to finish the problem.

$$F_y \;=\; x + 2z + \lambda xz \;=\; x + 2z - \frac{4xz}{x} \;=\; 0 \;\rightsquigarrow\; x + 2z - 4z \;=\; 0 \;\rightsquigarrow\; \boxed{x = 2z}$$

plug $x = y = 2z$ *into* $F_\lambda \;\rightsquigarrow\; xyz - 32 = 2z \cdot 2z \cdot z - 32 = 0 \;\rightsquigarrow\; z^3 \;=\; 8 \;\rightsquigarrow\; \boxed{z = 2}$

use again $x = y = 2z \;\rightsquigarrow\; \boxed{x = y = 4}$

The answer to the problem: Surface area is minimized if $x = y = 4$ feet and $z = 2$ feet.